HASSE PRINCIPLE AND BOMBIERI-MASSER-ZANNIER HEIGHT BOUND FOR THE DYNAMICS OF DIAGONALLY SPLIT POLYNOMIAL MAPS

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ABSTRACT. Let f be a polynomial of degree at least 2 with coefficients in a number field or a characteristic 0 function field K. In this paper, we study two arithmetic properties of the dynamics of the map $(f,...,f):\ (\mathbb{P}^1_K)^n \longrightarrow (\mathbb{P}^1_K)^n$ for $n\geq 2$, namely the dynamical analogues of the Hasse principle and the Bombieri-Masser-Zannier height bound theorem. In particular, we prove that the Hasse principle holds when we intersect an orbit and a preperiodic subvariety, and that points in the intersection of a curve with the union of all periodic hypersurfaces have bounded heights unless that curve is vertical or contained in a periodic hypersurface. An important common ingredient in our investigation of these two properties is the description of periodic subvarieties given recently by Medvedev and Scanlon.

1. INTRODUCTION

In algebraic dynamics, we are given a morphism φ from an algebraic variety X defined over a field K to itself, and our goal is to study the iterates φ^n and all the preperiodic subvarieties. The complex analytic side begins with work of Julia, Fatou and Ritt where $X=\mathbb{P}^1_{\mathbb{C}}$. The arithmetic side is more recent, starting with Northcott's paper [Nor50] in 1950 and, after a dormant period, advancing rapidly since the early 1990s. By analogies with well-known results in diophantine geometry, many results and conjectures in arithmetic dynamics have been formulated and partially verified, most notably the Uniform Boundedness Conjecture by Morton and Silverman [MS94], the Dynamical Mordell-Lang Conjecture by Ghioca and Tucker [GT09] and the Dynamical Manin-Mumford Conjecture by Ghioca, Tucker and Zhang [Zha06], [GTZ11].

In this paper, we study certain arithmetic properties of the dynamics of diagonally split polynomial maps $\varphi=(f,...,f):\ (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$ over a number field or a characteristic 0 function field K, where $n\geq 2$ and f is a polynomial of degree at least 2. There are a few good reasons why such maps deserve attention. First, we could study the dynamical analogues of many important diophantine results on the torus $\mathbb{G}_{\mathrm{m}}^n$ (for examples, see [Zan12]). Second, we could use a great deal of work by complex and arithmetic dynamists on polynomial maps of $\mathbb{P}^1$. Third, while the only subvarieties of $\mathbb{P}^1$ are points (and $\mathbb{P}^1$ itself), the variety $(\mathbb{P}^1)^n$ has many interesting positive dimensional subvarieties and provides a good testing ground for conjectures about dynamics of arbitrary varieties. Finally, very recently, by using

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model theory and Ritt's polynomial decomposition, Medvedev and Scanlon [MS13] are able to describe all the φ-periodic subvarieties of $(\mathbb{P}^1)^n$.

The arithmetic properties mentioned in the last paragraph are the dynamical Hasse principle, and the dynamical Bombieri-Masser-Zannier height bound theorem. For example, among our results are the following:

Theorem 1.1. *Let $n \geq 2$, let K be a number field or a characteristic 0 function field, f a polynomial of degree at least 2 in $K[X]$, and $\varphi = (f, ..., f) : (\mathbb{P}^1_K)^n \longrightarrow (\mathbb{P}^1_K)^n$. Let V be an absolutely irreducible preperiodic curve or hypersurface in $(\mathbb{P}^1_K)^n$, and $P \in (\mathbb{P}^1)^n(K)$ such that the φ-orbit of P does not intersect $V(K)$. If K is a function field, we assume f is not isotrivial (see Section 2). Then there are infinitely many primes $\mathfrak{p}$ of K such that the $\mathfrak{p}$-adic closure of the orbit of P does not intersect $V(K_{\mathfrak{p}})$, where $K_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic completion of K.*

Theorem 1.2. *Let K, n, f, and φ be as in Theorem 1.1, except that we even allow f to be isotrivial in the function field case. Assume f is disintegrated (see Definition 2.4). Let C be an irreducible curve in $(\mathbb{P}^1_{\bar{K}})^n$ that is not contained in any periodic hypersurface. Assume C maps surjectively onto each factor $\mathbb{P}^1$ of $(\mathbb{P}^1)^n$. Then points in*

$$\bigcup_V (C(\bar{K}) \cap V(\bar{K}))$$

have bounded heights, where V ranges over all periodic hypersurfaces of $(\mathbb{P}^1_{\bar{K}})^n$.

We refer the readers to Theorems 2.6, 2.7, 2.8, 3.4, 3.12, and 3.13 for much more general results. This paper consists of two parts: the dynamical Hasse principle is given in Section 2, while the dynamical Bombieri-Masser-Zannier height bound is given in Section 3. This section only serves as a brief introduction to these parts. Each part has its own introduction including motivation from diophantine geometry, and we refer the readers to those introductions for more details. We end this section by stating our convention for notation. A function field means a finitely generated field of transcendental degree 1 over a ground field of characteristic 0. Throughout this paper, K denotes a function field over the ground field κ or a number field, and M_K denotes the set of places of K. Note that in the function field case, by places of K, we mean the equivalence classes of the valuations on K that are trivial on κ. For every v in M_K, let K_v denote the completion of K with respect to v. If v is non-archimedean, we also let $\mathscr{O}_v$ and k_v respectively denote the valuation ring and the residue field of K_v. By a variety over K, we mean a reduced separated scheme of finite type over K. Every Zariski closed subset of a variety is identified with the corresponding induced reduced closed subscheme structure, and is called a closed subvariety. Curves, surfaces,..., and hypersurfaces are not assumed to be irreducible but merely *equidimensional*. In this paper, $\mathbb{P}^1_K$ is implicitly equipped with a coordinate function x having only one simple pole and zero which are denoted by ∞ and 0 respectively. Every polynomial $f \in K[X]$ gives a corresponding self-map of $\mathbb{P}^1_K$, and a self-map of $\mathbb{A}^1_K = \mathbb{P}^1_K - \{\infty\}$ by its action on x. For every self-map μ from a set to itself, for every positive integer n, we write μ^n to denote the n^{th} iterate of μ, and we define μ^0 to be the identity map. The phrase "for almost all" means "for all but finitely many".

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2. The Dynamical Hasse Principle

2.1. **Motivation and Main Results.** In this section, let S be a fixed finite subset of M_K containing all the archimedean places. For every variety X over K, we define:

$$X(K,S) = \prod_{v \notin S} X(K_v) \tag{1}$$

equipped with the product topology, where each $X(K_v)$ is given the v-adic topology which is Hausdorff by separatedness of X. The set $X(K)$ is embedded into $X(K,S)$ diagonally. For every subset T of $X(K,S)$, write $\mathcal{C}(T)$ to denote the closure of T in $X(K,S)$. The following theorem has been established by Poonen and Voloch [PV10, Theorem A]:

Theorem 2.1. *Assume that K is a function field. Let A be an abelian variety and V a subvariety of A both defined over K. Then:*

$$V(K) = V(K,S) \cap \mathcal{C}(A(K)). \tag{2}$$

The analogue of Theorem 2.1 when K is a number field is still widely open. The main motivation for Poonen-Voloch theorem is the determination of $V(K)$ especially when V is a curve of genus at least 2 embedded into its Jacobian. More precisely, they are interested in the Brauer-Manin obstruction to the Hasse principle previously studied by Scharaschkin, Skorobogatov, Flynn, Stoll... In fact, the idea of taking the (coarser) intersection between $V(K_{\mathfrak{p}})$ and the $\mathfrak{p}$-adic closure of $A(K)$ in $A(K_{\mathfrak{p}})$, where $\mathfrak{p}$ is a prime of K, is dated back to Chabauty's work in the 1940s. We refer the readers to [PV10] and the references there for more details.

Now return to our general setting, let φ be a K-morphism of X to itself, V a closed subvariety of X, and $P \in X(K)$ a K-rational point of X. We have the following inclusion (note the similarity with (2) where the finitely generated group $A(K)$ is replaced by the orbit of P):

$$V(K) \cap \mathcal{O}_\varphi(P) \subseteq V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)), \tag{3}$$

where $\mathcal{O}_\varphi(P) := \{P, \varphi(P), ...\}$ is the forward orbit of P.

Motivated by the Poonen-Voloch theorem, Hsia and Silverman [HS09, p. 237–238] ask:

Question 2.2. *Let V^{pp} denote the union of all positive dimensional preperiodic subvarieties of V. Assume that $\mathcal{O}_\varphi(P) \cap V^{\text{pp}}(K) = \emptyset$. When does equality hold in (3)?*

The requirement $\mathcal{O}_\varphi(P) \cap V^{\text{pp}}(K) = \emptyset$ is necessary as explained in [HS09, p. 238]. In this paper, we restrict to the following question:

Question 2.3. *Assume that V is preperiodic and $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$, when can we conclude $V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset$?*

Roughly speaking, while the Dynamical Mordell-Lang conjecture (DML) [GT09] involves the intersection of an orbit and a subvariety, Questions 2.2 and 2.3 involve the same kind of intersections but taken modulo ideals of $\mathscr{O}_{K,S}$ where $\mathscr{O}_{K,S}$ is the

ring of S-integers of K. Although the connection is not clear at the moment, one way to investigate Questions 2.2 and 2.3 is to look at cases where DML has been resolved. For instance, motivated by the solution of DML for étale maps by Bell, Ghioca and Tucker [BGT10], Amerik, Kurlberg, Towsley, Viray, Voloch and the author prove that if φ is étale, Question 2.3 has an affirmative answer [AKN$^+$, Section 4].

We remind the readers that if K is a function field over the constant field κ, a rational function $f \in K(X)$ is said to be isotrivial if there exists a fractional linear map $L \in \mathrm{Aut}(\mathbb{P}^1(\bar{K}))$ such that $L^{-1} \circ f \circ L \in \bar{\kappa}(X)$. We now introduce the notion of disintegrated polynomials. Let F be an algebraically closed field of characteristic 0. For $d \geq 2$, the Chebyshev polynomial of degree d is the unique polynomial $C_d \in F[X]$ such that $C_d(X + \dfrac{1}{X}) = X^d + \dfrac{1}{X^d}$.

Definition 2.4. *Let $f \in F[X]$ be a polynomial of degree $d \geq 2$. Then f is said to be special if there is $L \in \mathrm{Aut}(\mathbb{P}^1(F))$ such that $L^{-1} \circ f \circ L$ is either $\pm C_d$ or the power monomial X^d. The polynomial f is said to be disintegrated if it is not special.*

Here we have adopted the terminology "disintegrated polynomials" used in the Medvedev-Scanlon work [MS13] which has its origin from model theory. Unfortunately, there is no standard terminology for what we call special polynomials. Complex dynamists describe such maps as having "flat orbifold metric", Milnor [Mil] calls them "finite quotients of affine maps", and Silverman's book [Sil07] describes them as polynomials "associated to algebraic groups". The term "special" used here is succinct and sufficient for our purposes. We remark that for every $m > 0$, f^m is disintegrated if and only f is disintegrated. To prove this, we may assume $F = \mathbb{C}$ by the Lefschetz principle and use the fact that a polynomial is disintegrated if and only if its Julia set is not an interval or a circle.

Our main theorems below will address Question 2.3 when $X = (\mathbb{P}^1)^n$, and φ is the diagonally split morphism associated to a polynomial f. We begin with the case $\dim(V) = 0$:

Theorem 2.5. *Let $f \in K[X]$ be a polynomial of degree at least 2, let $n \geq 2$ be an integer, and let φ denote the split morphism $(f, \ldots, f)\colon (\mathbb{P}^1_K)^n \longrightarrow (\mathbb{P}^1_K)^n$. If K is a function field, we assume that f is not isotrivial. Let V be a zero dimensional subvariety of $(\mathbb{P}^1_K)^n$. The following hold:*

- (a) *For every $P \in (\mathbb{P}^1)^n(K)$ such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$, there exist infinitely many primes $\mathfrak{p}$ such that $V(K_\mathfrak{p})$ does not intersect the $\mathfrak{p}$-adic closure (in $(\mathbb{P}^1)^n(K_\mathfrak{p})$) of $\mathcal{O}_\varphi(P)$.*
- (b) *Question 2.2 has an affirmative answer, namely for every $P \in X(K)$ we have:*
$$V(K) \cap \mathcal{O}_\varphi(P) = V(K, S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)).$$
- (c) *In this part only, we assume f is **special** and V is **preperiodic**. Then for every $P \in (\mathbb{P}^1)^n(K)$ such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$, for **almost all** primes $\mathfrak{p}$ of K, we have $V(K_\mathfrak{p})$ does not intersect the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$.*

Part (b) actually holds for maps of the form $(f_1, \ldots, f_n)$ where each f_i is an arbitrary rational map of degree at least 2. This more general result follows from the main results of Silverman and Voloch [SV09]. We will see that the trick used

to establish part (a) in Subsection 2.2, which is similar to one used in [SV09], appears repeatedly in this section and can be modified to reduce our problem (when $\dim(V) > 0$) to the étale case (see Subsection 2.4). Part (c) of Theorem 2.5 could be generalized completely, we have:

Theorem 2.6. *Let $f \in K[X]$ be a **special** polynomial of degree $d \geq 2$. Let $n \geq 2$, and $\varphi = (f, \ldots, f)$ be as in Theorem 2.5. Let V be a subvariety of $(\mathbb{P}^1_K)^n$ such that every irreducible component of $V_{\bar{K}}$ is a preperiodic subvariety. Let $P \in (\mathbb{P}^1)^n(K)$ such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$. Then for **almost all** primes $\mathfrak{p}$ of K, $V(K_\mathfrak{p})$ does not intersect the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$. Consequently, Question 2.3 has an affirmative answer: $V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset$.*

It has been known since the beginning of the theory of complex dynamics that special polynomials and disintegrated polynomials have very different dynamical behaviours. When f is disintegrated, we are still able to prove that a Hasse principle analogous to Theorem 2.6 holds when V is a curve or a hypersurface:

Theorem 2.7. *Let $f \in K[X]$ be a **disintegrated** polynomial of degree $d \geq 2$. When K is a function field, we assume that f is not isotrivial. Let $n \geq 2$, and $\varphi = (f, \ldots, f)$ be as in Theorem 2.5. Let V be a φ-preperiodic and absolutely irreducible **curve** or **hypersurface** of $(\mathbb{P}^1_K)^n$. Let $P \in (\mathbb{P}^1)^n(K)$ such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$. Then for infinitely many primes $\mathfrak{p}$ of K, the $\mathfrak{p}$-adic closure of $\mathcal{O}_\mathfrak{p}(P)$ does not intersect $V(K_\mathfrak{p})$. Consequently, Question 2.3 has an affirmative answer: we have $V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset$.*

Although we expect Theorem 2.7 still holds for an arbitrary absolutely irreducible preperiodic subvariety V (i.e. $1 < \dim(V) < n-1$), we need to assume an extra technical assumption, as follows:

Theorem 2.8. *Let f, n, and φ be as in Theorem 2.7. Assume the technical assumption that every polynomial commuting with an iterate of f also commutes with f (this is satisfied by a generic f, see below). Let V be an absolutely irreducible φ-preperiodic subvariety of $(\mathbb{P}^1_K)^n$. Let $P \in (\mathbb{P}^1_K)^n$ such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$. Then there exist infinitely many primes $\mathfrak{p}$ of K such that the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$ does not intersect $V(K_\mathfrak{p})$. Consequently, Question 2.3 has an affirmative answer: $V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset$.*

In fact, let $M(f^\infty)$ denote the group of linear polynomials commuting with an iterate of f. By Proposition 2.14, if $M(f^\infty)$ is trivial then the technical assumption in Theorem 2.8 holds. When f has degree 2 and is not conjugate to X^2, we have that $M(f^\infty)$ is trivial. When f has degree at least 3, after making a linear change, we can assume:

$$f(x) = X^d + a_{d-2}X^{d-2} + a_{d-3}X^{d-3} + \ldots + a_0.$$

It is easy to prove that when $a_{d-2}a_{d-3} \neq 0$, the group $M(f^\infty)$ is trivial. Hence the technical assumption holds for a generic polynomial f.

When K is a number field, $\dim(V) = 0$ and V is preperiodic, Benedetto, Ghioca, Hutz, Kurlberg, Scanlon and Tucker [BGH+13, Theorem 1] have proved a much stronger result than part (a) of Theorem 2.5, namely there is a positive density set of primes $\mathfrak{p}$ such that the reduction mod $\mathfrak{p}$ of the orbit of P is still disjoint from the reduction mod $\mathfrak{p}$ of V. They also give a heuristic explanation why their result should not hold when $\dim(V)$ is large. In some sense, the results above might be

regarded as a common extension of results by Silverman and Voloch [SV09], and Benedetto, Ghioca, Hutz, Kurlberg, Scanlon and Tucker [BGH$^+$13] to the higher dimensional case.

The organization of this section is as follows. We first prove Theorem 2.5 which settles the case $\dim(V)=0$. After that, we prove Theorem 2.7 which establishes the case V is a curve or a hypersurface. Hence Theorem 2.8 holds when $n\in\{1,2,3\}$ or when $\dim(V)\in\{0,1,n-1\}$ without the extra technical assumption. To prove Theorem 2.8 for $n>2$, we proceed by induction on n. The same strategy also works for Theorem 2.6: we first consider the case V is a hypersurface, then proceed by induction.

For Theorem 2.7, by a result of Medvedev and Scanlon (see Theorem 2.13), we know that the curve or hypersurface V has a very special form. Then we use a result of Ingram-Silverman [IS09] to get infinitely many "good" primes. Let $\mathfrak{p}$ be such a prime, now under the assumption that $V(K_{\mathfrak{p}})$ is $\mathfrak{p}$-adically close to $\mathcal{O}_\varphi(P)$ and the special form of V, we can essentially reduce to the étale case treated in [AKN$^+$] and get the desired conclusion. For Theorem 2.6 when V is a hypersurface, we also reduce to the étale case. But the reduction steps will be much easier compared to those for Theorem 2.7. In the next subsection, we will give all the preliminary results alluded above as well as a proof of Theorem 2.5.

2.2. **An Assortment of Preliminary Results.** Our first lemma shows that in order to prove Theorems 2.5–2.8, we are free to replace K by a finite extension.

Lemma 2.9. *Let L be a finite extension of K, X a variety over K, φ a K-endomorphism of X, V a closed subvariety of X over K, and P an element of $X(K)$. Let $\mathfrak{p}$ be a prime of K and $\mathfrak{q}$ a prime of L lying above $\mathfrak{p}$. If $V(L_{\mathfrak{q}})$ does not intersect the $\mathfrak{q}$-adic closure of $\mathcal{O}_\varphi(P)$ in $X(L_{\mathfrak{q}})$ then $V(K_{\mathfrak{p}})$ does not intersect the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$ in $X(K_{\mathfrak{p}})$.*

Proof. Clear. □

Before stating the next result, we need some terminology. Let $\mathfrak{p}$ be a prime of K, $\mathscr{X}$ a separated scheme of finite type over $\mathscr{O}_{\mathfrak{p}}$. By the valuative criterion of separatedness [Har77, p. 97], we could view $\mathscr{X}(\mathscr{O}_{\mathfrak{p}})$ as a subset of of $\mathscr{X}(K_{\mathfrak{p}})$, then the $\mathfrak{p}$-adic topology on $\mathscr{X}(\mathscr{O}_{\mathfrak{p}})$ is the same as the subspace topology induced by the $\mathfrak{p}$-adic topology on $\mathscr{X}(K_{\mathfrak{p}})$. Every point $P\in\mathscr{X}(\mathscr{O}_{\mathfrak{p}})$ is an $\mathscr{O}_{\mathfrak{p}}$-morphism $\mathrm{Spec}(\mathscr{O}_{\mathfrak{p}})\longrightarrow\mathscr{X}$. By the generic point and closed point of P, we mean the image of the generic point and closed point of $\mathrm{Spec}(\mathscr{O}_{\mathfrak{p}})$, respectively. We write $\bar{P}$ to denote its closed point, which is also identified to the corresponding element in $\mathscr{X}(k_{\mathfrak{p}})$. The scheme $\mathscr{X}$ is said to be smooth at P if the structural morphism $\mathscr{X}\longrightarrow\mathrm{Spec}(\mathscr{O}_{\mathfrak{p}})$ is smooth at $\bar{P}$. Similarly, an endomorphism φ of $\mathscr{X}$ over $\mathscr{O}_{\mathfrak{p}}$ is said to be étale at P if it is étale at $\bar{P}$. The following is essentially a main result of [AKN$^+$, Theorem 4.4]:

Theorem 2.10. *Let K, $\mathfrak{p}$, $\mathscr{X}$, and $P\in\mathscr{X}(\mathscr{O}_{\mathfrak{p}})$ be as in the last paragraph. Let φ be an endomorphism of $\mathscr{X}$ over $\mathscr{O}_{\mathfrak{p}}$. Assume that $\mathscr{X}$ is smooth and φ is étale at every point in the orbit $\mathcal{O}_\varphi(P)$. Let $\mathscr{V}$ be a reduced closed subscheme of $\mathscr{X}$. Assume one of the following sets of conditions:*

(a) *There exists $M>0$ satisfying $\varphi^M(\mathscr{V})\subseteq\mathscr{V}$. When K is a function field, we assume that P is φ-preperiodic modulo $\mathfrak{p}$ (this condition is automatic for number fields since $k_{\mathfrak{p}}$ is finite).*

(b) *$\mathscr{V}$ is a finite set of preperiodic points of $\mathscr{X}(\mathscr{O}_{\mathfrak{p}})$.*

We have: if $\mathscr{V}(\mathscr{O}_{\mathfrak{p}})$ does not intersect $\mathcal{O}_{\varphi}(P)$ then it does not intersect the $\mathfrak{p}$-adic closure of $\mathcal{O}_{\varphi}(P)$.

Proof. First assume the conditions in (a). Although the statement in [AKN$^+$, Theorem 4.4] includes smoothness of $\mathscr{X}$ and étaleness of φ everywhere, its proof could actually be carried verbatim here.

Now assume the conditions in (b). Define:

$$\mathscr{V}_1 = \bigcup_{i=0}^{\infty} \varphi^i(\mathscr{V}).$$

Then $\mathscr{V}_1$ is a finite set of points in $\mathscr{X}(\mathscr{O}_{\mathfrak{p}})$ satisfying $\varphi(\mathscr{V}_1) \subseteq \mathscr{V}_1$. If the orbit of P intersects $\mathscr{V}_1(\mathscr{O}_{\mathfrak{p}})$ then P is preperiodic and there is nothing to prove. So we may assume otherwise. After reducing mod $\mathfrak{p}$, if the orbit of P does not intersect $\mathscr{V}_1$ then there is nothing to prove. So we may assume otherwise, and this assumption gives that P is preperiodic mod $\mathfrak{p}$. All the conditions in part (a) are now satisfied, and we can get the desired conclusion. □

The Silverman-Voloch trick mentioned right after Theorem 2.5 is the following (see [IS09] for all the terminology):

Lemma 2.11. *Let $f \in K[X]$ be a polynomial of degree at least 2, and let $\alpha \in K$ be an f-wandering point. Assume f is not isotrivial if K is a function field. Let $\gamma \in K$ be a periodic point of f such that f is not of polynomial type at γ (this always holds if the exact period of γ is at least 3* [IS09, p. 291]*). Then there are infinitely many primes $\mathfrak{p}$ of K such that $v_{\mathfrak{p}}(f^{\mu}(\alpha) - \gamma) > 0$ for some μ depending on $\mathfrak{p}$.*

Proof. This follows from the deeper result of Ingram-Silverman [IS09, p. 292] that almost all elements of the sequence $(f^{\mu}(\alpha) - \gamma)$ have a primitive divisor. Or we could make a fractional linear change of variable and use the finiteness of integral points in orbits proved in [Sil93, Theorem B]. □

We can use Lemma 2.11 to prove the following:

Lemma 2.12. *Let f be as in Lemma 2.11. Let α be an element of $\mathbb{P}^1(K)$ and V a finite subset of $\mathbb{P}^1(K)$ such that the orbit of α does not intersect V. Then there are infinitely many primes $\mathfrak{p}$ such that the $\mathfrak{p}$-adic closure of the orbit of α does not intersect V.*

Proof. If α is preperiodic, there is nothing to prove (actually every prime $\mathfrak{p}$ will satisfy the desired conclusion). We now assume α is wandering. For almost all $\mathfrak{p}$, we have $v_{\mathfrak{p}}(\alpha) \geq 0$ and $f \in \mathscr{O}_{\mathfrak{p}}[X]$, so $v_{\mathfrak{p}}(f^m(\alpha)) \geq 0$ for all m. Therefore we can assume $\infty \notin V$. Let $u_1, ..., u_q$ be all elements of V. By Lemma 2.9, we can assume there is a periodic point $\gamma \in K$ of exact period at least 3 and the orbit of γ does not contain u_i for $1 \leq i \leq q$. Hence there is a finite set of primes T such that:

$$f \in \mathscr{O}_{\mathfrak{p}}[X] \text{ and } v_{\mathfrak{p}}(u_i - f^m(\gamma)) = 0 \ \forall m \geq 0 \ \forall 1 \leq i \leq q \ \forall \mathfrak{p} \notin T. \tag{4}$$

Now by Lemma 2.11, there are infinitely many primes $\mathfrak{p} \notin T$ such that:

$$v_{\mathfrak{p}}(f^{\mu}(\alpha) - \gamma) > 0 \text{ for some } \mu = \mu_{\mathfrak{p}} \tag{5}$$

Fix any $\mathfrak{p} \notin T$ that gives (5), write $\mu = \mu_{\mathfrak{p}}$. Thus $v_{\mathfrak{p}}(f^m(\alpha) - f^{m-\mu}(\gamma)) > 0$ for every $m \geq \mu$. Together with (4), we have $v_{\mathfrak{p}}(f^m(\alpha) - u_i) = 0$ for every $m \geq \mu$ and

$1 \leq i \leq q$. This implies that V does not intersect the $\mathfrak{p}$-adic closure of the f-orbit of α. $\square$

Now we have all the results needed to prove Theorem 2.5:
Proof of Theorem 2.5: By Lemma 2.9, we can replace K by a finite extension so that V is a finite set of points in $(\mathbb{P}^1)^n(K)$.

For part (a), note that if P is φ-preperiodic then there is nothing to prove, hence we can assume P is wandering. Write $P = (\alpha_1, \ldots, \alpha_n)$, without loss of generality, we assume α_1 is wandering with respect to f. Let U denote the finite subset of $\mathbb{P}^1(K)$ consisting of the first coordinates of points in V. There is the largest N such that $f^N(\alpha) \in U$. We simply replace P by $\varphi^{N+1}(\alpha)$ and assume the f-orbit of α does not contain any element of U. Then our conclusion follows from Lemma 2.12.

Part (b) follows easily from part (a). As before, we can assume P is not preperiodic, hence there is the largest N such that $\varphi^N(P) \in V(K)$. Replacing P by $\varphi^{N+1}(P)$, we can assume that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$, then part (a) implies

$$V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset = V(K) \cap \mathcal{O}_\varphi(P).$$

For part (c), we first consider the case $L \circ f \circ L^{-1} = X^d$ for some linear polynomial $L \in \bar{K}[X]$. By extending K, we may assume $L \in K[X]$. Since L yields a homeomorphism from $\mathbb{P}^1(K_\mathfrak{p})$ to itself for almost all $\mathfrak{p}$, we may assume $f(X) = X^d$. As before, we can assume the first coordinate α_1 of P is wandering and its orbit does not contain any element of U. This latter condition is actually redundant since V is φ-preperiodic, hence U only consists of f-preperiodic points. For almost all $\mathfrak{p}$, the first coordinates of points in the φ-orbit of P is a $\mathfrak{p}$-adic unit. Therefore we can exclude from V all the points having first coordinates 0 or ∞, hence $U \subseteq \mathbb{G}_\mathrm{m}(K)$. Let $\mathfrak{p}$ be a prime not dividing d such that α_1 and all elements of U are $\mathfrak{p}$-adic units. We now apply Theorem 2.10 for $\mathscr{X} = \mathbb{G}_\mathrm{m}$ over $\mathscr{O}_\mathfrak{p}$, $\mathscr{V} = U \subseteq \mathbb{G}_\mathrm{m}(\mathscr{O}_\mathfrak{p})$, the self-map being the multiplication-by-d map, and the orbit of α_1. Since the $\mathfrak{p}$-adic closure of the orbit of α_1 does not intersect $U(K_\mathfrak{p})$, the $\mathfrak{p}$-adic closure of P does not intersect $V(K_\mathfrak{p})$.

For the case $L \circ f \circ L^{-1} = \pm C_d(X)$, we use the self-map of $(\mathbb{P}^1)^n$ given by:

$$(x_1, \ldots, x_n) \mapsto (x_1 + \frac{1}{x_1}, \ldots, x_n + \frac{1}{x_n})$$

to reduce to the case that f is conjugate to $\pm X^d$ which has just been treated. This finishes the proof of Theorem 2.5.

2.3. **The Medvedev-Scanlon Theorem.** Throughout this subsection, let F be an algebraically closed field of characteristic 0, and $n \geq 2$ a positive integer. Based on earlier work of Chatzidakis, Hrushovski and Peterzil [CHP02], Medvedev [Med] shows that for a rational map $f \in F(X)$ of degree $d \geq 2$, the associated self-map of $\mathbb{P}^1_F$ is disintegrated if and only if f is not conjugate to X^d, $\pm C_d(X)$ or Lattès. For the notion of a disintegrated self-map of a variety over F, we refer the readers to [MS13]. Roughly speaking, a self-map of $\mathbb{P}^1_F$ is disintegrated if every periodic subvariety of $(\mathbb{P}^1_F)^n$ under the corresponding split self-map is given by equations each of which only involves two variables. When f is a polynomial, Ritt's theory of polynomial decomposition allows one to describe all the periodic curves in $(\mathbb{P}^1_F)^2$ under (f, f). This leads to the following theorem of Medvedev-Scanlon [MS13, p. 5] which is a crucial ingredient in our paper:

Theorem 2.13. *Let $f \in F[X]$ be a disintegrated polynomial of degree $d \geq 2$, let $n \geq 2$ and let $\varphi = (f, ..., f) : (\mathbb{P}^1_F)^n \longrightarrow (\mathbb{P}^1_F)^n$. Let V be an irreducible φ-invariant (respectively φ-periodic) subvariety in $(\mathbb{P}^1_F)^n$. For $1 \leq i \leq n$, let x_i be the chosen coordinate for the i^{th} factor of $(\mathbb{P}^1)^n$ (see the last paragraph of Section 1). Then V is given by a collection of equations of the following types:*

(A) *$x_i = \zeta$ where ζ is a fixed (respectively periodic) point of f.*
(B) *$x_j = g(x_i)$ for some $i \neq j$, where g is a polynomial commuting with f (respectively an iterate of f).*

We could further describe all the polynomials g in type (B) of Theorem 2.13 as follows.

Proposition 2.14. *Let F and f be as in Theorem 2.13. We have:*

(a) *If $g \in F[X]$ has degree at least 2 such that g commutes with an iterate of f then g and f have a common iterate.*
(b) *Let $M(f^\infty)$ denote the collection of all linear polynomials commuting with an iterate of f. Then $M(f^\infty)$ is a finite cyclic group under composition.*
(c) *Let $\tilde{f} \in F[X]$ be a polynomial of lowest degree at least 2 such that $\tilde{f}$ commutes with an iterate of f. Then there exists $D = D_f > 0$ relatively prime to the order of $M(f^\infty)$ such that $\tilde{f} \circ L = L^D \circ \tilde{f}$ for every $L \in M(f^\infty)$.*
(d) *$\left\{\tilde{f}^m \circ L : \ m \geq 0, \ L \in M(f^\infty)\right\} = \left\{L \circ \tilde{f}^m : \ m \geq 0, \ L \in M(f^\infty)\right\}$, and these sets describe exactly all polynomials g commuting with an iterate of f.*

Proof. By Lefschetz principle, we may assume $F = \mathbb{C}$. Part (a) is a well-known result of Ritt [Rit23, p. 399]. For part (b), let Σ_f denote the group of linear fractional automorphism of the Julia set of f. It is known that Σ_f is finite cyclic [SS95]. Therefore $M(f^\infty)$, being a subgroup of Σ_f is also finite cyclic. By part (a), f and $\tilde{f}$ have the same Julia set. Therefore $\Sigma_f = \Sigma_{\tilde{f}}$. By [SS95], there exists D such that $\tilde{f} \circ L = L^D \circ \tilde{f}$ for every $L \in \Sigma_f = \Sigma_{\tilde{f}}$. To prove that D is relatively prime to the order of $M(f^\infty)$, we let $\tilde{L}$ denote a generator of $M(f^\infty)$, and $N > 0$ such that $\tilde{L} \circ \tilde{f}^N = \tilde{f}^N \circ L$. Hence $\tilde{L} \circ \tilde{f}^N = \tilde{L}^{D^N} \circ \tilde{f}^N$. The last equality implies $D^N - 1$ is divisible by the order of $M(f^\infty)$ and we are done.

It remains to show part (d). The given two sets are equal since D^m is relatively prime to the order of $M(f^\infty)$ for every $m \geq 0$. It suffices to show if $g \in F[X]$, $\deg(g) > 1$ and g commutes with f then g has the form $\tilde{f}^m \circ L$. Let $\varphi = (f, f)$ be the split self-map of $(\mathbb{P}^1_F)^2$. Now the (possibly reducible) curve V in $(\mathbb{P}^1_F)^2$ given by $\tilde{f}(y) = g(x)$ satisfies $\varphi^M(V) \subseteq V$ for some $M > 0$. Therefore some irreducible component C of V is periodic. By Theorem 2.13, C is given by $y = \psi(x)$ or $x = \psi(y)$ where ψ commutes with an iterate of f. Therefore one of the following holds:

(i) $\tilde{f} \circ \psi = g$
(ii) $g \circ \psi = \tilde{f}$

Since $\deg(g) \geq \deg(\tilde{f})$ by the definition of $\tilde{f}$, case (ii) can only happen when $\deg(g) = \deg(\tilde{f})$ and $\psi \in M(f^\infty)$. If this is the case, we can write (ii) into $g = \tilde{f} \circ (\psi)^{-1}$. Thus we can assume (i) always happens. Repeating the argument for the pair $(\tilde{f}, \psi)$ instead of $(\tilde{f}, g)$, we get the desired conclusion. □

Remark 2.15. Proposition 2.14 follows readily from Ritt's theory of polynomial decomposition. The proof given here uses the Medvedev-Scanlon description in Theorem 2.13 and simple results from complex dynamics. In fact, in an upcoming work, we will study and give examples of a lot of rational (and non-polynomial) maps f such that Theorem 2.13 is still valid. Then an analogue of Proposition 2.14 (especially part (d)) still holds by exactly the same proof.

We conclude this section with a particularly useful property of preperiodic subvarieties of $(\mathbb{P}^1_F)^n$. Let f, n and φ be as in Theorem 2.13. Let V be an irreducible φ-periodic subvariety of $(\mathbb{P}^1_F)^n$. We will associate to V a binary relation $\prec$ on $I = \{1, \ldots, n\}$ as follows. Let I_V denote the set of $1 \leq i \leq n$ such that V is contained in a hypersurface of the form $x_i = \zeta$ where ζ is a periodic point. The relation $\prec$ is empty if and only if $I_V = I$ (i.e. V is a point). For every $i \in I - I_V$, we include the relation $i \prec i$. For two elements $i \neq j$ in $I - I_V$, we include the relation $i \prec j$ if V is contained in a hypersurface of the form $x_j = g(x_i)$ where g is a polynomial commuting with an iterate of f. We have the following properties:

Lemma 2.16. *Notations as in the last paragraph. Let $1 \leq i, j, k \leq n$. We have:*

(a) *Transitivity: if $i \prec j$ and $j \prec k$ then $i \prec k$.*
(b) *Upper chain extension: if $i \prec j$ and $i \prec k$ then either $j \prec k$ or $k \prec j$.*
(c) *Lower chain extension: if $i \prec k$ and $j \prec k$ then either $i \prec j$ or $j \prec i$.*

Proof. We may assume i, j, and k are distinct, otherwise there is nothing to prove. Part (a) is immediate from the definition of $\prec$. For part (b), we have that V is contained in hypersurfaces $x_j = g_1(x_i)$ and $x_k = g_2(x_i)$. By Proposition 2.14, we may write $g_1 = g_3 \circ g_2$ or $g_2 = g_3 \circ g_1$ for some g_3 commuting with an iterate of f. This implies $k \prec j$ or $j \prec k$.

Now we prove part (c). Let π denote the projection from $(\mathbb{P}^1)^n$ onto the (i, j, k)-factor $(\mathbb{P}^1)^3$. We have that $\pi(V)$ is an irreducible (f, f, f)-periodic curve of $(\mathbb{P}^1)^3$ contained in the (not necessarily irreducible) curve given by $x_k = g_1(x_i)$ and $x_k = g_2(x_j)$ (note that we must have $\dim(\pi(V)) > 0$ since $i, j, k \notin I_V$). Now we consider the closed embedding:

$$(\mathbb{P}^1_F)^2 \xrightarrow{\eta} (\mathbb{P}^1_F)^3$$

defined by $\eta(y_i, y_j) = (y_i, y_j, g_1(y_i))$. Now $\eta^{-1}(\pi(V))$ is an irreducible (f, f)-periodic curve of $(\mathbb{P}^1_F)^2$ whose projection to each factor $\mathbb{P}_1$ is surjective since $i, j \notin I_V$. Therefore $\eta^{-1}(\pi(V))$ is given by either $y_i = g_3(y_j)$ or $y_j = g_3(y_i)$ for some g_3 commuting with an iterate of f. This implies either $j \prec i$ or $i \prec j$. $\square$

A chain is either a tuple of one element (i) where $i \notin I_V$ (equivalently $i \prec i$), or an ordered set of distinct elements $i_1 \prec i_2 \prec \ldots \prec i_l$. If $\mathcal{I} = (i_1, \ldots, i_l)$ is a chain, we denote the underlying set (or the support) $\{i_1, \ldots, i_l\}$ by $s(\mathcal{I})$. Note that it is possible for many chains to have a common support, for example if V is contained in $x_j = g(x_i)$ where g is linear then both (i, j) and (j, i) are chains. By Lemma 2.16, if $\mathcal{I}$ is a chain, $i \in I$ and $i \prec j$ or $j \prec i$ for some $j \in \mathcal{I}$ then we can enlarge $\mathcal{I}$ into a chain whose support is $s(\mathcal{I}) \cup \{i\}$. We have that there exist maximal chains $\mathcal{I}_1, \ldots, \mathcal{I}_l$ whose supports partition $I - I_V$. Although the collection $\{\mathcal{I}_1, \ldots, \mathcal{I}_l\}$ is not uniquely determined by V, the collection of supports $\{s(\mathcal{I}_1), \ldots, s(\mathcal{I}_l)\}$ is. To prove these facts, one may define an equivalence relation $\approx$ on $I - I_V$ by $i \approx j$ if and only if $i \prec j$ or $j \prec i$. Then it is easy to prove that $\{s(\mathcal{I}_1), \ldots, s(\mathcal{I}_l)\}$ is exactly the collection of equivalence classes.

For an ordered subset J of I, we define the following factor of $(\mathbb{P}^1)^n$:

$$(\mathbb{P}^1)^J :=\prod_{j\in J}\mathbb{P}^1.$$

For a collection of ordered sets $J_1,\ldots,J_l$ whose underlying (i.e. unordered) sets partition I, we have the canonical isomorphism:

$$(\mathbb{P}^1)^n = (\mathbb{P}^1)^{J_1}\times\ldots\times(\mathbb{P}^1)^{J_l}.$$

We now have the following result:

Proposition 2.17. *Let f and φ be as in Theorem 2.13. Let V be an irreducible φ-preperiodic subvariety of $(\mathbb{P}^1_F)^n$. Assume that $\dim(V)>0$. Let $I=\{1,\ldots,n\}$, and let I_V denote the set of all i's such that V is contained in a hypersurface of the form $x_i=\zeta_i$ where ζ_i is f-preperiodic. We fix a choice of an order on I_V, write $l=\dim(V)$. There exist a collection of ordered sets $J_1,\ldots,J_l$ whose underlying sets partition $I-I_V$ such that under the canonical isomorphism*

$$(\mathbb{P}^1)^n = (\mathbb{P}^1)^{I_V}\times(\mathbb{P}^1)^{J_1}\times\ldots\times(\mathbb{P}^1)^{J_l},$$

we have:

$$V=(\prod_{i\in I_V}\{\zeta_i\})\times V_1\times\ldots\times V_l$$

where V_k is an $(f,\ldots,f)$-preperiodic curve of $(\mathbb{P}^1)^{J_k}$ for $1\le k\le l$.

Proof. There exists m such that $\varphi^m(V)$ is periodic. The conclusion of the Proposition for $\varphi^m(V)$ will imply the same conclusion for V, hence we may assume V is periodic. We associate to V a binary relation $\prec$ on I as before. Then there exist maximal chains $\mathcal{I}_1,\ldots,\mathcal{I}_l$ whose supports partition $I-I_V$. We now take $J_k=\mathcal{I}_k$ for $1\le k\le l$. □

2.4. **Proof of Theorem 2.7.** Let $f\in K[X]$ be a disintegrated polynomial of degree $d\ge 2$. By Theorem 2.13, for every preperiodic hypersurface H of $(\mathbb{P}^1_K)^n$, there exist $1\le i<j\le n$ such that $H=\pi^{-1}(C)$ where π denotes the projection onto the (i,j)-factor and C is an (f,f)-preperiodic curve of $(\mathbb{P}^1_K)^2$. Therefore it suffices to prove Theorem 2.7 when V is a curve.

Let φ, V and $P\in(\mathbb{P}^1)^n(K)$ such that $V(K)\cap\mathcal{O}_\varphi(P)=\emptyset$ and $\dim(V)=1$ as in Theorem 2.7 and the discussion in the last paragraph. Let I and I_V be as in Proposition 2.17. Let $\tilde{f}$, $M(f^\infty)$, and $D=D_f$ be as in Proposition 2.14. Now we prove that there are infinitely many primes $\mathfrak{p}$ of K such that $V(K_\mathfrak{p})$ does not intersect the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$. By Lemma 2.9, we can assume $\tilde{f}$ and all elements in $M(f^\infty)$ have coefficients in K.

Step 1: We first consider the case V is periodic.

Step 1.1: we assume that $I_V=\emptyset$. By Theorem 2.13, Proposition 2.17 and the discussion before it, we can relabel the factors of $(\mathbb{P}^1)^n$ and rename the coordinate functions of all the factors as x, $y_1,\ldots,y_{n-1}$ such that V is given by the equations: $y_i=g_i(x)$ for $1\le i\le n-1$, where g_i commutes with an iterate of f for $1\le i\le n-1$ and $\deg(g_1)\le\ldots\le\deg(g_{n-1})$. Write $P=(a,b_1,\ldots,b_{n-1})$.

Step 1.1.1: we consider the easy case that a is f-preperiodic. Replacing P by an iterate, we can assume that a is f-periodic of exact period N. The φ-orbit of P is:

$$\{(f^i(a),f^{i+tN}(b_1),\ldots,f^{i+tN}(b_{n-1})):\ t\ge 0,\ 0\le i<N\}.$$

Since this orbit does not intersect $V(K)$, we have

$$\forall t \geq 0\ \forall 0 \leq i < N\ \exists 1 \leq j \leq n-1\ \left(f^{i+tN}(b_j) \neq g_j(f^i(a))\right). \tag{6}$$

For each $0 \leq i < N$ and $1 \leq j \leq n-1$, denote $\mathcal{B}_{i,j} = (f^i)^{-1}(\{g_j(f^i(a))\})$. Denote

$$\mathcal{B} = \bigcup_{0 \leq i < N} \mathcal{B}_{i,1} \times \ldots \times \mathcal{B}_{i,n-1}$$

which is a finite set of (preperiodic) points of $(\mathbb{P}^1)^{n-1}$. Let $b = (b_1, \ldots, b_{n-1})$ and let ϕ denote the self-map $(f, \ldots, f)$ of $(\mathbb{P}^1)^{n-1}$. By (6), we have $\phi^{tN}(b) \notin \mathcal{B}$ for every $t \geq 0$. By Theorem 2.5, there exist infinitely many primes $\mathfrak{p}$ such that the $\mathfrak{p}$-adic closure $\mathcal{C}_{\mathfrak{p}}$ of $\{\phi^{tN}(b) :\ t \geq 0\}$ does not intersect $\mathcal{B}$. For each such $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of the orbit of P lies in:

$$\bigcup_{0 \leq i < N} \{f^i(a)\} \times \phi^i(C_{\mathfrak{p}})$$

which is disjoint from $V(K_{\mathfrak{p}})$.

Step 1.1.2: we turn to the most difficult case, namely a is a wandering point of f. By Proposition 2.14, write $f = \rho_1 \circ \tilde{f}^A$ for some integer $A \geq 1$ and linear $\rho_1 \in M(f^\infty)$. Since D is relatively prime to $|M(f^\infty)|$, there is $\rho_2 \in M(f^\infty)$ such that $f = (\tilde{f} \circ \rho_2)^A$. Replacing $\tilde{f}$ by $\tilde{f} \circ \rho_2$, we may assume $f = \tilde{f}^A$. For each $1 \leq j \leq n-1$, write $g_j = L_j \circ \tilde{f}^{m_j}$.

For almost all $\mathfrak{p}$, we have $v_{\mathfrak{p}}(f^n(a)) \geq 0$ for every $n \geq 0$. If for some $1 \leq j \leq n-1$, $b_j = \infty$ then for almost all $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of the orbit of P lies in:

$$\{(x, y_1, \ldots, y_{j-1}, \infty, y_{j+1}, \ldots, y_{n-1}) :\ x \in K_{\mathfrak{p}},\ v_{\mathfrak{p}}(x) \geq 0\}$$

which is disjoint from $V(K_{\mathfrak{p}})$. So we can assume $b_j \neq \infty$ for every $1 \leq j \leq n-1$.

By taking a finite extension of K if necessary, we choose an $\tilde{f}$-periodic point $\gamma \in K$ of exact period $N \geq 3$ such that every point in the $\tilde{f}$-orbit of γ is not a zero of the derivative $\tilde{f}'(X)$ of $\tilde{f}(X)$. By Lemma 2.11, there is an infinite set of primes R such that for every $\mathfrak{p} \in R$, all of the following hold:

$$a, b_1, \ldots, b_{n-1} \in \mathscr{O}_{\mathfrak{p}}, \text{ in other words } P \in \mathbb{A}^n(\mathscr{O}_{\mathfrak{p}}) \tag{7}$$

$$v_{\mathfrak{p}}(\tilde{f}'(\tilde{f}^i(\gamma))) = 0\ \forall\ 0 \leq i < N \tag{8}$$

(9) $\tilde{f}, f, L_1, \ldots, L_{n-1}$ are in $\mathscr{O}_{\mathfrak{p}}[X]$ and their leading coefficients are $\mathfrak{p}$-adic units

$$v_{\mathfrak{p}}(\tilde{f}^\mu(a) - \gamma) > 0 \text{ for some } \mu = \mu_{\mathfrak{p}}. \tag{10}$$

In fact, for $1 \leq j \leq n-1$, the leading coefficient c_j of L_j is a root of unity and $f = \tilde{f}^A$ is an iterate of $\tilde{f}$, hence the conditions on c_j and f in (9) are redundant. Now fix a prime $\mathfrak{p}$ in R and write $\mu = \mu_{\mathfrak{p}}$, we still use V to denote the model $y_j = L_j \circ \tilde{f}^{m_j}(x) \forall j$ over $\mathscr{O}_{\mathfrak{p}}$, hence it makes sense to write $V(\mathscr{O}_{\mathfrak{p}})$ and $V(k_{\mathfrak{p}})$. We also use P, and φ to denote the corresponding models over $\mathscr{O}_{\mathfrak{p}}$. Replacing P by $\varphi^\mu(P)$ and γ by $\tilde{f}^{A\mu-\mu}(\gamma)$ if necessary, we can assume that $v_{\mathfrak{p}}(a - \gamma) > 0$. This gives that a is $\tilde{f}$-periodic, hence f-periodic, modulo $\mathfrak{p}$ and:

$$v_{\mathfrak{p}}(\tilde{f}^l(a) - \tilde{f}^l(\gamma)) > 0 \text{ and } v_{\mathfrak{p}}(\tilde{f}'(\tilde{f}^l(a)) - \tilde{f}'(\tilde{f}^l(\gamma))) > 0\ \forall l \geq 0 \tag{11}$$

The second inequality in (11) together with (8) give:

$$v_{\mathfrak{p}}(\tilde{f}'(\tilde{f}^l(a))) = 0\ \forall l \geq 0 \tag{12}$$

By (12) and induction, we have:

$$v_{\mathfrak{p}}((\tilde{f}^l)'(\tilde{f}^k(a))) = 0 \ \forall l \geq 0 \ \forall k \geq 0 \tag{13}$$

Since $f = \tilde{f}^A$, identity (13) implies:

$$v_{\mathfrak{p}}((f^l)'(f^k(a))) = 0 \ \forall l \geq 0 \ \forall k \geq 0 \tag{14}$$

Since the φ-orbit of P lies in $\mathbb{A}^n(\mathscr{O}_{\mathfrak{p}})$ which is closed in $(\mathbb{P}^1)^n(K_{\mathfrak{p}})$, it suffices to show that $V(\mathscr{O}_{\mathfrak{p}})$ does not intersect the $\mathfrak{p}$-adic closure of the φ-orbit of P. Assume there is η such that the mod $\mathfrak{p}$ reduction $\varphi^\eta(\bar{P})$ lies in $V(k_{\mathfrak{p}})$, otherwise there is nothing to prove. After replacing P by $\varphi^\eta(P)$, we can assume $\eta = 0$, or in other words $\bar{P} \in V(k_{\mathfrak{p}})$. This means

$$v_{\mathfrak{p}}(b_j - L_j \circ \tilde{f}^{m_j}(a)) > 0 \ \forall 1 \leq j \leq n-1. \tag{15}$$

Note that $\tilde{f} \circ L = L^D \circ \tilde{f}$, and L has finite order, therefore (15) together with the $\tilde{f}$-periodicity mod $\mathfrak{p}$ of a give that b_j is $\tilde{f}$-preperiodic, hence f-preperiodic, mod $\mathfrak{p}$ for $1 \leq j \leq n-1$. Therefore P is φ-preperiodic mod $\mathfrak{p}$.

Inequality (15) shows that:

$$v_{\mathfrak{p}}(\tilde{f}^l(b_j) - L_j^{D^l} \circ \tilde{f}^{l+m_j}(a)) > 0 \ \forall l \geq 0 \ \forall 1 \leq j \leq n-1. \tag{16}$$

Our next step is to show:

$$v_{\mathfrak{p}}(\tilde{f}'(L_j^{D^l} \circ \tilde{f}^{l+m_j}(a))) = 0 \ \forall l \geq 0 \ \forall 1 \leq j \leq n-1 \tag{17}$$

Recall that c_j denotes the leading coefficient of the linear polynomial L_j, we have:

$$(\tilde{f} \circ L_j^{D^l} \circ \tilde{f}^{m_j+l})'(a) = (L_j^{D^{l+1}} \circ \tilde{f}^{m_j+l+1})'(a) = c_j^{D^{l+1}}(\tilde{f}^{m_j+l+1})'(a) \tag{18}$$

and

$$(\tilde{f} \circ L_j^{D^l} \circ \tilde{f}^{m_j+l})'(a) = \tilde{f}'(L_j^{D^l} \circ \tilde{f}^{m_j+l}(a)) c_j^{D^l}(\tilde{f}^{m_j+l})'(\alpha) \tag{19}$$

Since c_j is a $\mathfrak{p}$-adic unit, (18) and (19) imply:

$$v_{\mathfrak{p}}((\tilde{f}^{m_j+l+1})'(a)) = v_{\mathfrak{p}}(\tilde{f}'(L_j^{D^l} \circ \tilde{f}^{m_j+l}(a))(\tilde{f}^{m_j+l})'(\alpha)) \tag{20}$$

Now (17) follows from (13), and (20). By (16) and (17), we have:

$$v_{\mathfrak{p}}(\tilde{f}'(\tilde{f}^l(b_j))) = 0 \ \forall l \geq 0 \ \forall 1 \leq j \leq n-1 \tag{21}$$

By (21) and induction, we have:

$$v_{\mathfrak{p}}((\tilde{f}^l)'(\tilde{f}^k(b_j))) = 0 \ \forall l \geq 0 \ \forall k \geq 0 \ \forall 1 \leq j \leq n-1 \tag{22}$$

Since $f = \tilde{f}^A$, identity (22) implies:

$$v_{\mathfrak{p}}((f^l)'(f^k(b_j))) = 0 \ \forall l \geq 0 \ \forall k \geq 0 \ \forall 1 \leq j \leq n-1 \tag{23}$$

Now (14) and (23) show that the $\mathscr{O}_{\mathfrak{p}}$-morphism φ is étale at every $\mathscr{O}_{\mathfrak{p}}$-valued point in the orbit of P. Together with the fact that P is preperiodic mod $\mathfrak{p}$, we can apply Theorem 2.10 to get the desired conclusion. This finishes the case V is periodic and $I_V = \emptyset$.

Step 1.2: assume that $I_V \neq \emptyset$. Let π and π' denote the projection from $(\mathbb{P}^1_K)^n$ onto $(\mathbb{P}^1)^{I_V}$ and $(\mathbb{P}^1)^{I-I_V}$, respectively. By Proposition 2.17, $V = \pi_1(V) \times \pi_2(V)$ where $Z := \pi_1(V)$ is a periodic point of $(\mathbb{P}^1)^{I_V}$. Write $W = \pi_2(V)$. By Step 1.1, the conclusion of Theorem 2.7 is valid for W. Let φ_1 and φ_2 respectively denote the diagonally split self-map of $(\mathbb{P}^1)^{I_V}$ and $(\mathbb{P}^1)^{I-I_V}$ associated to f.

Step 1.2.1: assume there is the largest N such that $\varphi_1^N(\pi_1(P)) = Z$. Replace P by $\varphi^{N+1}(P)$, we can assume that the φ_1-orbit of $\pi_1(P)$ does not contain Z. By Theorem 2.5, there exist infinitely many primes $\mathfrak{p}$ such that the $\mathfrak{p}$-adic closure $\mathcal{C}_{\mathfrak{p}}$ of $\mathcal{O}_{\varphi_1}(\pi_1(P))$ does not contain Z. For each such $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$ is contained in $\mathcal{C}_{\mathfrak{p}} \times (\mathbb{P}^1)^{I-I_V}(K_{\mathfrak{p}})$ which is disjoint from $V(K_{\mathfrak{p}})$.

Step 1.2.2: now assume $\varphi_1^n(\pi_1(P)) = Z$ for infinitely many n. This implies that Z is periodic and $\pi_1(P)$ is preperiodic. Replacing P by an iterate, we may assume $\pi_1(P) = Z$. Let N denote the exact period of Z. Since $\mathcal{O}_\varphi(P)$ does not intersect V, we have that $\mathcal{O}_{\varphi_2^N}(\pi_2(P))$ does not intersect W. By Step 1.1, the theorem holds for W. Hence there exist infinitely many primes $\mathfrak{p}$ such that the $\mathfrak{p}$-adic closure $\mathcal{C}_{\mathfrak{p}}$ of $\mathcal{O}_{\varphi_2^N}(\pi_2(P))$ does not intersect $W(K_{\mathfrak{p}})$. For each such $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$ is contained in:

$$\left(\bigcup_{i=1}^{N-1} \{\varphi_1^i(Z)\} \times (\mathbb{P}^1)^{I-I_V}(K_{\mathfrak{p}})\right) \cup \{Z\} \times \mathcal{C}_{\mathfrak{p}}$$

which is disjoint from $V(K_{\mathfrak{p}})$. This finishes the case V is periodic.

Step 2: assume V is preperiodic and not periodic, hence there exist $k > 0$ and $M > 0$ such that $\varphi^{k+M}(V) = \varphi^k(V)$. For $0 \le i < M$, write $V_i = \varphi^{k+i}(V)$. Then we have that V_i is periodic for every $0 \le i < M$.

Step 2.1: assume $I_V = \emptyset$. As in Step 1.1, we can relabel the factors of $(\mathbb{P}^1)^n$ and rename the coordinate functions into $x, y_1, \ldots, y_{n-1}$ so that for each $0 \le i < M$, the periodic curve V_i is given by equations $y_j = g_{i,j}(x)$ for $1 \le j \le n-1$, where $g_{i,j}$ commutes with an iterate of f and $\deg(g_{i,1}) \le \ldots \le \deg(g_{i,n-1})$.

Since V is not periodic, V and V_i are distinct curves, hence $V \cap V_i$ is a finite set of points for every $0 \le i < M$. By Lemma 2.9, we extend K such that $\mathbb{P}^1(K)$ contains the coordinates of all these points.

Now we assume that for almost all $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$ intersects $V(K_{\mathfrak{p}})$ and we will arrive at a contradiction. Because $V_0 = \varphi^k(V)$, for every such $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$ intersects $V_0(K_{\mathfrak{p}})$. Since V_0 is periodic, the conclusion of Theorem 2.7 has been established for V_0. We must have that $V_0(K)$ contains an element in the orbit of P. By ignoring the first finitely many elements in that orbit, we may assume $P \in V_0(K)$. Then we have $\varphi^{i+tM}(P) \in V_i(K)$ for all $t \ge 0$, $0 \le i < M$. Let a denote the x-coordinate of P. For each $0 \le i < M$, let $n_i = |V \cap V_i|$, and let $u_{i,1}, ..., u_{i,n_i}$ denote the x-coordinates of points in $V \cap V_i$. Since V_i is defined by $y_j = g_i(x)$ for $1 \le j \le n-1$, every point on V_i is uniquely determined by its x-coordinate. Since the orbit of P does not intersect $V(K)$, we have:

$$f^{i+tM}(a) \notin \{u_{i,1}, \ldots, u_{i,n_i}\} \ \forall t \ge 0, \ \forall 0 \le i < M.$$

Write $\mathcal{A} = \bigcup_{0 \le i < M} (f^i)^{-1}(\{u_{i,1}, \ldots, u_{i,n_i}\})$. We have that $f^{tM}(a) \notin \mathcal{A}$ for all $t \ge 0$. By Theorem 2.5, there exists infinitely many primes $\mathfrak{q}$ such that the $\mathfrak{q}$-adic closure $C_{\mathfrak{q}}$ of $\{f^{tM}(a) : \ t \ge 0\}$ does not intersect $\mathcal{A}$. Now the $\mathfrak{q}$-adic closure of the orbit of P is contained in:

$$\bigcup_{0 \le i < M} \left(V_i(K_{\mathfrak{q}}) \cap \{(x, y_1, \ldots, y_{n-1}) \in (\mathbb{P}^1)^n(K_{\mathfrak{q}}) : \ x \in f^i(C_{\mathfrak{q}})\}\right)$$

which is disjoint from $V(K_{\mathfrak{q}})$. This gives a contradiction and finishes the case $I_V = \emptyset$.

Step 2.2: assume $I_V \neq \emptyset$. We can reduce to Step 2.1 in exactly the same way we reduce Step 1.2 to Step 1.1. This finishes the proof of Theorem 2.7.

2.5. **Proof of Theorem 2.8.** In this subsection, we prove Theorem 2.8 by using induction on n. The cases $n \in \{1, 2, 3\}$ or $\dim(V) \in \{0, 1, n-1\}$ have been established by Theorem 2.5 and Theorem 2.7 even without the extra technical assumption of Theorem 2.8. Now assume $N > 3$ and Theorem 2.8 holds for all $1 \leq n < N$, we consider the case $n = N$. We may assume $\dim(V) > 1$. As in the proof of Theorem 2.7 in the last subsection, we can assume $I_V = \emptyset$ and deduce the more general case in exactly the same way.

Step 1: assume V is periodic. By Theorem 2.13, there exist $1 \leq i < j \leq n$ such that the image $\pi(V)$ of V under the projection

$$\pi\colon (\mathbb{P}^1)^n \to (\mathbb{P}^1)^2$$

onto the (i,j)-factor is a periodic curve. We may assume $(i,j) = (1,2)$. If $\pi(\mathcal{O}_\varphi(P))$ does not intersect $\pi(V)(K)$ then we can apply the induction hypothesis. Otherwise, by ignoring the first finitely many elements in the orbit of P, we may assume $\pi(P) \in \pi(V)(K)$.

Since $I_V = \emptyset$, we may assume $\pi(V)$ is given by the equation $x_2 = g(x_1)$ where g commutes with an iterate of f (the case $x_1 = g(x_2)$ is similar). Our technical assumption gives that g commutes with f. We consider the closed embedding:

$$(\mathbb{P}^1)^{n-1} \xrightarrow{e} (\mathbb{P}^1)^n$$

defined by $e(y_1, \ldots, y_{n-1}) = (y_1, g(y_1), y_2, \ldots, y_{n-1})$. By pulling back under e, we reduce our problem to the subvariety $(\mathbb{P}^1)^{n-1}$ and apply the induction hypothesis. This finishes the case V is periodic.

Step 2: assume V is preperiodic and not periodic. Write $\delta = \dim(V)$. By Proposition 2.17 and without loss of generality, there exist $m_0 = 0 < m_1 < m_2 < \ldots < m_\delta = n$ such that $V = C_1 \times C_2 \times \ldots \times C_\delta$ where each C_i is an $(f, \ldots, f)$-preperiodic curve of $(\mathbb{P}^1)^{m_i - m_{i-1}}$ for $1 \leq i \leq \delta$. For $1 \leq i \leq \delta$, let π_i denote the corresponding projection from $(\mathbb{P}^1)^n$ onto $(\mathbb{P}^1)^{m_i - m_{i-1}}$, and let φ_i denote the self-map $(f, \ldots, f)$ of $(\mathbb{P}^1)^{m_i - m_{i-1}}$. If P is preperiodic then there is nothing to prove, hence we may assume P is wandering. Without loss of generality, we assume $\pi_1(P)$ is φ_1-wandering.

Step 2.1: assume C_1 is not φ-periodic (recall that it is preperiodic). Then the set

$$\bigcup_{j>0} C_1 \cap \varphi_1^j(C_1)$$

is finite. Since $\pi_1(P)$ is wandering, there are only finitely many j's such that $\varphi_1^j(\pi_1(P))$ is contained in $C_1(K)$. Ignore finitely many points in the orbits of P, we may assume that the φ_1-orbit of $\pi_1(P)$ does not intersect $C_1(K)$. Then we can apply the induction hypothesis for the data $((\mathbb{P}^1)^{m_1}, \varphi_1, \pi_1(P), C_1)$.

Step 2.2: assume C_1 is φ-periodic. If the φ_1-orbit of $\pi_1(P)$ does not intersect $C_1(K)$ then we can apply the induction hypothesis as above. So we may assume some element in this orbit is in $C_1(K)$. Replacing P by an iterate, we may assume $\pi_1(P) \in C_1(K)$. Since $I_V = \emptyset$, the curve C_1 is not contained in any hypersurface of

the form $x_j = \gamma$. By Proposition 2.17 and the discussion before it, we know that C_1 is either $\mathbb{P}^1$ if $m_1 = 1$ or is given by equations of the form (after possibly relabeling the variables $x_1, \ldots, x_{m_1}$): $x_2 = g_1(x_1)$, $x_3 = g_2(x_1)$, ..., $x_{m_1} = g_{m_1-1}(x_{m_1-1})$, where each g_j commutes with an iterate of f. By our technical assumption, every g_j commutes with f. Hence C_1 is φ_1-invariant, and we have $\pi_1(\varphi^l(P)) \in C_1(K)$ for every $l \geq 0$. Let P' denote the image of P under the projection from $(\mathbb{P}^1)^n$ to $(\mathbb{P}^1)^{m_2-m_1} \times \ldots \times (\mathbb{P}^1)^{m_\delta - m_{\delta-1}}$. We now apply the induction hypothesis for the data:

$$((\mathbb{P}^1)^{n-m_1}, \varphi_2 \times \ldots \times \varphi_\delta, C_2 \times \ldots \times C_\delta, P').$$

This finishes the proof of Theorem 2.8.

2.6. **Proof of Theorem 2.6 when V is a hypersurface.** We first consider the case $\sigma \circ f \circ \sigma^{-1} = X^d$ for some $\sigma \in \mathrm{Aut}(\mathbb{P}^1)$. By extending K, we may assume $\sigma \in K[X]$. For almost all $\mathfrak{p}$, σ induces a homeomorphism from $(\mathbb{P}^1)^n(K_\mathfrak{p})$ to itself. Hence we can assume $f(X) = X^d$. Since the conclusion of Theorem 2.6 is for almost all $\mathfrak{p}$, we can assume V is an absolutely irreducible preperiodic hypersurface defined over K.

First, assume there exists $1 \leq i \leq n$ such that V is given by $x_i = 0$ or $x_i = \infty$. By the automorphism $X \mapsto X^{-1}$ and without loss of generality, we may assume V is given by $x_1 = 0$. Let α denote the first coordinate of P, since the orbit of P does not intersect $V(K)$, we have $\alpha \neq 0$. For almost all $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of the orbit of P lies in:

$$\{(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n(K_\mathfrak{p}) : \ v_p(x_1) = 0\}$$

which is disjoint from $V(K_\mathfrak{p})$.

Therefore, we may assume $V \cap \mathbb{G}_\mathrm{m}^n \neq \emptyset$. It is not difficult to prove that $V \cap \mathbb{G}_\mathrm{m}^n$ is a translate of a subgroup of codimension 1, see [Zan12, Remark 1.1.1]. We now denote the coordinate of each factor $\mathbb{P}^1$ as $x_1, ..., x_q$, $y_1, ..., y_r$ and $z_1, ..., z_s$ (hence $q + r + s = n$) such that V is given by an equation:

$$x_1^{a_1} ... x_q^{a_q} = \zeta y_1^{b_1} .... y_r^{b_r},$$

where $a_1, ..., b_r$ are positive integers, and ζ is a root of unity. Actually, for V to be preperiodic, we have $\zeta^{d^A} = \zeta^{d^B}$ for some $0 \leq A < B$; but we will not need this stronger fact. Write P under the corresponding coordinates as:

$$P = (\alpha_1, ..., \alpha_q, \beta_1, ..., \beta_r, \gamma_1, ..., \gamma_s).$$

Assume some elements among the $\alpha_1, ..., \beta_r$ are either 0 or ∞, say, we have $\alpha_1 = 0$. Then each irreducible component of the intersection $V \cap \{x_1 = 0\}$ has the form $\{x_1 = 0 \wedge x_i = \infty\}$ for $2 \leq i \leq q$, or the form $\{x_1 = 0 \wedge y_j = 0\}$ for some $1 \leq j \leq r$. Thus the coordinates of P satisfy:

$$(\alpha_i \neq \infty \ \forall 2 \leq i \leq q) \wedge (\beta_j \neq 0 \ \forall 1 \leq j \leq r).$$

For every prime $\mathfrak{p}$, let $v_\mathfrak{p}(\infty) = -\infty$ (warning: the ∞ on the left is an element of $\mathbb{P}^1(K)$ while the ∞ on the right is an element of the extended real numbers). For almost all primes $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of the orbit of P is contained in:

$$\{(0, X_2, ..., Z_s) : \ v_\mathfrak{p}(X_i) \geq 0, \ v_\mathfrak{p}(Y_j) \leq 0 \ \forall 2 \leq i \leq q \ \forall 1 \leq j \leq r\}$$

which is disjoint from $V(K_\mathfrak{p})$. The case, say, $\alpha_1 = \infty$ is treated similarly.

Now we can assume that all the $\alpha_1, ..., \beta_r$ lie in $\mathbb{G}_{\mathrm{m}}(K)$. Let

$$\eta = \alpha_1^{a_1} ... \alpha_q^{a_q} \beta_1^{-b_1} ... \beta_r^{-b_r}.$$

Since the φ-orbit of P does not intersect $V(K)$, we have that the f-orbit of η does not contain ζ. For almost all $\mathfrak{p}$, we have η is a $\mathfrak{p}$-adic unit. By Theorem 2.10, for almost all $\mathfrak{p}$, the $\mathfrak{p}$-adic closure $C_{\mathfrak{p}}(\eta)$ of the orbit of η does not contain ζ. Now the $\mathfrak{p}$-adic closure of the orbit of P lies in

$$\left\{ (X_1, ..., Z_s) : \; X_1^{a_1} ... Y_r^{-b_r} \in C_{\mathfrak{p}}(\eta) \right\}$$

which is disjoint from $V(K_{\mathfrak{p}})$. This finishes the case f is a conjugate of X^d.

Now we assume f is a conjugate of $\pm C_d(X)$. As before, we may assume $f(X) = \pm C_d(X)$. Let $\hat{f} = \pm X^d$, and $\hat{\varphi}$ be the diagonally split morphism corresponding $\hat{f}$. Consider the morphisms:

$$\Phi \colon (x_1, ..., x_n) \mapsto \left(x_1 + \frac{1}{x_1}, ..., x_n + \frac{1}{x_n} \right)$$

from $(\mathbb{P}^1)^n$ to itself. We have the commutative diagram:

$$\begin{tikzcd}
(\mathbb{P}^1)^n \arrow[r, "\hat{\varphi}"] \arrow[d, "\Phi"'] & (\mathbb{P}^1)^n \arrow[d, "\Phi"] \\
(\mathbb{P}^1)^n \arrow[r, "\varphi"'] & (\mathbb{P}^1)^n
\end{tikzcd}$$

Extend K further, we may assume there is $Q \in (\mathbb{P}^1)^n(K)$ such that $\Phi(Q) = P$. Write $\hat{V} = \Phi^{-1}(V)$. We have that the Φ-orbit of Q does not intersect $\hat{V}(K)$. Note that the conclusion of the theorem has been established for $\hat{f}$. Therefore, for almost all $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of $\mathcal{O}_{\hat{\varphi}}(Q)$ does not intersect $\hat{V}(K_{\mathfrak{p}})$. Since Φ is finite, it maps the $\mathfrak{p}$-adic closure of $\mathcal{O}_{\hat{\varphi}}(Q)$ onto the $\mathfrak{p}$-adic closure of $\mathcal{O}_{\varphi}(P)$. We can conclude that the $\mathfrak{p}$-adic closure of $\mathcal{O}_{\varphi}(P)$ does not intersect $V(K_{\mathfrak{p}})$.

2.7. **Proof of Theorem 2.6.** As in Subsection 2.6, we first consider the case f is conjugate to X^d, and then we may assume $f(X) = X^d$. By Theorem 2.5 and Subsection 2.6, we have that Theorem 2.6 is valid when $n = 1, 2$. We proceed by induction on n. Let $N \geq 3$ and assume that Theorem 2.6 holds for all $n < N$, we now consider the case $n = N$. As in Subsection 2.6, we assume V is an absolutely irreducible preperiodic subvariety defined over K.

We first consider the case V is contained in a hypersurface of the form $x_i = 0$ or $x_i = \infty$ for some $1 \leq i \leq n$. Without loss of generality, we may assume V is contained in the hypersurface $x_1 = 0$. Let α denote the first coordinate of P. If $\alpha \neq 0$ then for almost all $\mathfrak{p}$, $\mathfrak{p}$-adic closure of the orbit of P is contained in:

$$\{(x_1, \ldots, x_n) : \; v_{\mathfrak{p}}(x_1) = 0\}$$

which is disjoint from $V(K_{\mathfrak{p}})$. Hence we assume $\alpha = 0$. We now restrict to the hyperplane $x_1 = 0$ and apply the induction hypothesis.

Therefore we may assume $V \cap \mathbb{G}_{\mathrm{m}}^n \neq \emptyset$. Write $P = (\alpha_1, \ldots, \alpha_n)$. We first consider the case $P \notin \mathbb{G}_{\mathrm{m}}^n$. Without loss of generality, assume $\alpha_1 = 0$. We can again restrict to the hypersurface $x_1 = 0$ and apply the induction hypothesis.

Now consider the case $P \in \mathbb{G}_{\mathrm{m}}^n$. For almost all $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of the orbit of P lies in:

$$(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n(K_{\mathfrak{p}}) : \ v_p(x_i) = 0 \ \forall 1 \leq i \leq n\}$$

which is closed in both $(\mathbb{P}^1)^n(K_{\mathfrak{p}})$ and $\mathbb{G}_{\mathrm{m}}^n(K_{\mathfrak{p}})$. Hence it suffices to show that for almost all $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of $\mathcal{O}_\varphi(P)$ in $\mathbb{G}_{\mathrm{m}}^n(K_{\mathfrak{p}})$ does not intersect $(V \cap \mathbb{G}_{\mathrm{m}}^n)(K_{\mathfrak{p}})$. This follows from the main result of [AKN$^+$, Theorem 4.3].

3. Dynamical Bombieri-Masser-Zannier Height Bound

3.1. **Motivation and Main Results.** Our original motivation comes from the following:

Question 3.1 (Lang, Manin-Mumford)**.** *Let X be an abelian variety or the torus $\mathbb{G}_m^n$ over $\mathbb{C}$. Let C be an irreducible curve in X. Assume C is not a torsion translate of a subgroup. Is it true that there are only finitely many torsion points on C?*

This question has an affirmative answer. When X is an abelian variety, it is a special case of the Manin-Mumford conjecture first proved by Raynaud [Ray83]. When $X = \mathbb{G}_{\mathrm{m}}^n$, it is a special case of a question of Lang stated in the 1960s (see, for example, [Lan83]) which admits many proofs as well as generalizations. A naive dynamical analogue of Lang's question for the dynamics of split polynomial maps could be the following:

Question 3.2. *Let $f \in \mathbb{C}[X]$ be a polynomial of degree at least 2, and let $\varphi : (\mathbb{P}^1_{\mathbb{C}})^2 \longrightarrow (\mathbb{P}^1_{\mathbb{C}})^2$ be the corresponding split polynomial maps. Let C be an irreducible curve in $(\mathbb{P}^1_{\mathbb{C}})^2$ such that C is not preperiodic. Is it true that C can only contain finitely many preperiodic points?*

The full dynamical analog of the Manin-Mumford conjecture (Raynaud's theorem) has been proposed by Zhang [Zha06] and Zhang-Ghioca-Tucker [GTZ11]. By using Zhang's method [Zha92], Mimar [Mim13] shows that Question 3.2 has an affirmative answer when C is the graph of a function from $\mathbb{P}^1$ to itself.

On the other hand, among vast generalizations of Lang's question, we have the following theorem of Bombieri, Masser and Zannier [BMZ99, Theorem 1]:

Theorem 3.3 (Bombieri, Masser, Zannier)**.** *Let C be an irreducible curve in $\mathbb{G}_m^n$ defined over a number field K such that C is not contained in any translate of a proper subgroup. Then*

(a) *Points in $\bigcup_V (C(\bar{K}) \cap V(\bar{K}))$ have bounded height, where V ranges over all subgroup of codimension 1.*
(b) *The set $\bigcup_V (C(\bar{K}) \cap V(\bar{K}))$ is finite, where V ranges over all subgroups of codimension 2.*

Questions 3.1, 3.2 and part (b) of Theorem 3.3 are instances of unlikely intersection. Very roughly speaking, when two objects that are not expected to intersect happen to intersect "a lot", then there has to be an explanation coming from the geometric structure of these objects. For more details, including an extensive survey of recent developments on unlikely intersection in diophantine geometry, we refer the readers to Zannier's book [Zan12] and the references there. Our main result is the dynamical analog of part (a) of Theorem 3.3 for the dynamics of split polynomial maps:

Theorem 3.4. *Let K be a number field or a function field. Let $f \in K[X]$ be a disintegrated polynomial, and $\varphi : (\mathbb{P}^1_K)^n \longrightarrow (\mathbb{P}^1_K)^n$ be the corresponding split polynomial map. Let C be an irreducible curve in $(\mathbb{P}^1_{\bar{K}})^n$ that is not contained in any periodic hypersurface. Assume C is non-vertical, by which we mean C maps surjectively onto each factor $\mathbb{P}^1$ of $(\mathbb{P}^1)^n$. Then the points in*

$$\bigcup_V (C(\bar{K}) \cap V(\bar{K}))$$

have bounded Weil heights, where V ranges over all periodic hypersurfaces of $(\mathbb{P}^1_K)^n$.

Recall our convention that although a hypersurface is not necessarily irreducible, it is assumed to be equidimensional. Hence it makes no difference if we impose the extra condition that V is irreducible in Theorem 3.4.

On the one hand, the conclusion of Theorem 3.4 is an interesting result on its own. On the other hand, note that Bombieri, Masser and Zannier prove part (b) of Theorem 3.3 by using part (a) combined with certain results involving the Lehmer's conjecture [BMZ99]. Therefore, one might hope to somehow use results like Theorem 3.4 as a first step in the resolution of Question 3.2.

We expect Theorem 3.4 still holds in the non-preperiodic case: C is assumed to be not contained in any preperiodic hypersurface, and V ranges over all preperiodic hypersurfaces. However, we could only prove a bound on the "average height" of points in the intersections (see Theorem 3.12). In fact, such bound on the average height turns out to hold for a more general polarized dynamical system (see Theorem 3.13). We prove this general result by using various constructions of heights and canonical heights coming from Gillet-Soulé generalization of Arakelov intersection theory (see [BGS94], [Zha95], and [Kaw06]).

In fact, part (a) of Theorem 3.3 is only the beginning of a long and unfinished story. Subsequent papers by various authors have considered bounded height results for higher dimensional complementary intersections in the torus $\mathbb{G}_{\mathrm{m}}^n$ or an abelian variety. We refer the readers to [BMZ07], [Hab08], and [Hab09] as well as the references there for more details. As far as we know, the results given in this section are the first to indicate that the above results in diophantine geometry are expected to hold, at least to some extent, in arithmetic dynamics. We will treat the dynamical analogue of higher dimensional complementary intersections in a future work. In this paper, we will be content with intersection between a curve and preperiodic hypersurfaces.

Throughout this section, let $f \in K[X]$ be a disintegrated polynomial. We use h to denote the absolute logarithmic Weil height on $\mathbb{P}^1(\bar{\mathbb{Q}})$. We also use h to denote the height on $(\mathbb{P}^1)^n(\bar{\mathbb{Q}})$ defined by $h(a_1, \ldots, a_n) = h(a_1) + \ldots + h(a_n)$. For every polynomial $P \in \bar{K}[X]$ of degree at least 2, we let $\hat{h}_P$ denote the canonical height associated to P. We use $\hat{h}$ to denote the canonical height $\hat{h}_f$. For properties of all these height functions, see [HS00, Part B] and [Sil07, Chapter 3].

3.2. **Proof of Theorem 3.4.** Since the projection from C to each factor $\mathbb{P}^1$ is finite, to show that a collection of points in $C(\bar{K})$ has bounded heights, it suffices to show that for some $1 \leq i \leq n$, all their x_i-coordinates have bounded heights. By the Medvedev-Scanlon Theorem, it suffices to show that for every $1 \leq i < j \leq n$, points in $\bigcup C(\bar{K}) \cap V_{ij}(\bar{K})$ have bounded heights where V_{ij} ranges over all periodic

hypersufaces whose equation involving x_i and x_j only. Therefore we may assume $n = 2$ for the rest of this subsection. Let x and y denote the coordinate functions on the first and second factor $\mathbb{P}^1$ respectively. Without loss of generality, we only need to consider the intersection with periodic curves V given by an equation of the form $x = \zeta$ where ζ is f-periodic, or $y = g(x)$ where g commutes with an iterate of f. Now every periodic ζ has height bounded uniformly, we get the desired conclusion when intersecting C with curves of the form $x = \zeta$. Note that this argument also works for all preperiodic ζ.

So we only have to consider curves V of the form $y = g(x)$. Let (M, N) denote the type of the divisor C of $(\mathbb{P}^1)^2$. Explicitly, we choose a generator $F(x, y)$ of the (prime) ideal of C in $\bar{K}[x, y]$, then F has degree M in x and degree N in y. We have the following two easy lemmas:

Lemma 3.5. *For every point (α, β) in $C(\bar{K})$, we have:*

$$|M\hat{h}(\alpha) - N\hat{h}(\beta)| \leq c_1\sqrt{\hat{h}(\alpha) + \hat{h}(\beta) + 1} + c_2. \tag{24}$$

where c_1 and c_2 are constants independent of (α, β).

Proof. Let $\tilde{C}$ denote the normalization of C, we have:

$$\tilde{C} \xrightarrow{\eta} C \xrightarrow{i} (\mathbb{P}^1_K)^2 \tag{25}$$

where η is the normalization map and i is the closed embedding realizing C as a subvariety of $(\mathbb{P}^1_K)^2$.The invertible sheaf $\mathscr{L} := (\eta \circ i)^*\mathscr{O}(1, 1)$ is ample on $\tilde{C}$. Let π_1 and π_2 denote respectively the first and second projections from $(\mathbb{P}^1_K)^2$ to $\mathbb{P}^1_K$. The invertible sheaves $\mathscr{L}_1 := (\eta \circ i \circ \pi_1)^*\mathscr{O}(1)$ and $\mathscr{L}_2 := (\eta \circ i \circ \pi_2)^*\mathscr{O}(1)$ have degrees N and M, respectively.

For $j = 1, 2$, define $\tilde{h}_j(P) = h(\pi_j \circ i \circ \eta(P))$ for every $P \in \tilde{C}(K)$. We also define $\tilde{h}(P) = h(i \circ \eta(P))$ for every $P \in \tilde{C}(K)$. Then $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ respectively are height functions on $\tilde{C}(K)$ corresponding $\mathscr{L}$, $\mathscr{L}_1$ and $\mathscr{L}_2$. By [HS00, Theorem B.5.9], there is a constant $c_1 > 0$ depending only on the data (25) such that:

$$|M\tilde{h}_1(P) - N\tilde{h}_2(P)| \leq c_1\sqrt{\tilde{h}(P) + 1} \quad \forall P \in \tilde{C}(K) \tag{26}$$

For every point $(\alpha, \beta) \in C(K)$, inequality (26) gives:

$$|Mh(\alpha) - Nh(\beta)| \leq c_1\sqrt{h(\alpha) + h(\beta) + 1}. \tag{27}$$

In term of the canonical height function associated to f, inequality (27) becomes:

$$|M\hat{h}(\alpha) - N\hat{h}(\beta)| \leq c_1\sqrt{\hat{h}(\alpha) + \hat{h}(\beta) + 1} + c_2 \tag{28}$$

where c_2 only depends on f and the data (25). □

Lemma 3.6. *Let $P \in \bar{K}[X]$ be a disintegrated polynomial, $\mathscr{G}$ a finite cyclic subgroup of linear polynomials in $\bar{K}[X]$ such that for some positive integer D, we have $P \circ L = L^D \circ P$ for every $L \in \mathscr{G}$. We have:*

(a) $\hat{h}_P = \hat{h}_{L\circ P^l}$ *for every $l > 0$ and every $L \in \mathscr{G}$.*
(b) $\hat{h}_P(L(\alpha)) = \hat{h}_P(\alpha)$ *for every $L \in \mathscr{G}$ and $\alpha \in \mathbb{P}^1(\bar{K})$.*

Proof. Since $\mathscr{G}$ is finite, there is ϵ such that:

$$h(x) - \epsilon \leq h(L(x)) \leq h(x) + \epsilon \ \forall x \in \bar{K} \ \forall L \in \mathscr{G}.$$

For every $k \geq 1$, we have $(L \circ P^l)^k = \tilde{L} \circ P^{kl}$ for some $\tilde{L} \in \mathscr{G}$. And we have:

$$h((L \circ P^l)^k(x)) = h(P^{kl}(x)) + O(1)$$

where $O(1)$ is bounded independently of k. Dividing both sides by $\deg(P^{kl})$ and let $k \to \infty$ will kill off this $O(1)$. Part (b) is proved similarly. □

We can now finish the proof of Theorem 3.4. Let V be given by $y = g(x)$ and (α, β) be a point in the intersection $C \cap V$. By Lemma 3.6, we have $\hat{h}(\beta) = \deg(g)\hat{h}(\alpha)$. Substituting this into (24), we have:

$$|M - N\deg(g)|\hat{h}(\alpha) \leq c_1\sqrt{(\deg(g)+1)\hat{h}(\alpha)+1} + c_2 \tag{29}$$

For all sufficiently large $\deg(g)$ (for instance, we may choose $\deg(g) > \dfrac{2M}{N}$ so that $N\deg(g) - M > \dfrac{N\deg(g)}{2}$), inequality (29) implies that $\hat{h}(\alpha)$ and hence $h(\alpha)$ is bounded above by a constant depending only on f and the data (25). Therefore by the remark at the first paragraph of this subsection, $h(\alpha, \beta)$ is bounded by a constant depending only on f and the data (25). Finally, by Proposition 2.14, there are only finitely many such $g's$ of bounded degree, hence only finitely many points in the intersection $C \cap \{y = g(x)\}$. This finishes the proof of Theorem 3.4.

3.3. **Further Questions.** We now gather several questions concerning the union $\bigcup_V (C(\bar{K}) \cap V(\bar{K}))$ where V ranges over preperiodic hypersurfaces in $(\mathbb{P}^1_{\bar{K}})^n$ and C is not contained in any such hypersurface. For each $k \geq 0$, let $\mathscr{P}_k$ denote the collection of all hypersurfaces V of $(\mathbb{P}^1_{\bar{K}})^n$ such that $\varphi^k(V)$ is periodic. Thus $\mathscr{P}_0$ is exactly the collection of periodic hypersurfaces, and we have $\mathscr{P}_k \subseteq \mathscr{P}_{k+1}$ for every k. Apply Theorem 3.4 for $\varphi^k(C)$, let Γ_k denote an upper bound for the f-canonical heights of points in

$$\bigcup_{V \in \mathscr{P}_0} (\varphi^k(C)(\bar{K}) \cap V(\bar{K})).$$

Using

$$\varphi^k(\bigcup_{V \in \mathscr{P}_k} (C(\bar{K}) \cap V(\bar{K}))) \subseteq \bigcup_{V \in \mathscr{P}_0} (\varphi^k(C)(\bar{K}) \cap V(\bar{K})),$$

we have that points in $\bigcup_{V \in \mathscr{P}_k} (C(\bar{K}) \cap V(\bar{K}))$ have canonical heights bounded by $\dfrac{\Gamma_k}{d^k}$ where $d \geq 2$ is the degree of f. Heuristically speaking, suppose we could obtain a bound in Theorem 3.4 that depends, in a uniform way, on the "complexity" of C, and the "complexity" of $\varphi^k(C)$ is "essentially" the "complexity" of C multiplied by d^k. Then we have that $\Gamma_k = d^k O(1)$ where $O(1)$ is independent of k. All of these motivate the following questions. From now on, we assume K is a number field although the first two questions could be asked for function fields as well:

Question 3.7. *Let f and φ be as in Theorem 3.4.*

(a) *Let C be an irreducible non-vertical curve in $(\mathbb{P}^1_{\bar{K}})^n$. Suppose C is not contained in an element of $\mathscr{P}_k$. Is it true that points in*

$$\bigcup_{V \in \mathscr{P}_k} (C(\bar{K}) \cap V(\bar{K}))$$

have heights bounded independently of k.

(b) *Let C be an irreducible non-vertical curve in $(\mathbb{P}^1_{\bar{K}})^n$ that is not contained in any preperiodic hypersurface. Is it true that points in*

$$\bigcup_V (C(\bar{K}) \cap V(\bar{K}))$$

have bounded heights, where V ranges over all preperiodic hypersurfaces of $(\mathbb{P}^1_{\bar{K}})^n$?

(c) *Let C be as in part (b). Is it true that the union in (b) have only finitely points of bounded degree?*

(d) *Let C be as in (b). Assume C is defined over K. Is it true that the union in (b) have only finitely many K-rational points?*

It is obvious that these questions have decreasing strength. From now on, we will focus on Question 3.7(b). We now look more closely to the proof of Theorem 3.4 and see what still go through. Assume f, φ and C as in part (b) of Question 3.7. As before, we can assume V ranges over all irreducible preperiodic hypersurfaces. Let $k \geq 0$ such that $\varphi^k(V)$ is periodic, hence given by an equation of the form, say, $x_j = g(x_i)$ where $1 \leq i < j \leq n$ (the case $\varphi^k(V)$ is given by $x_i = \zeta$ where ζ is preperiodic is easy). We can now assume $n = 2$ by projecting to the (i, j)-factor $(\mathbb{P}^1)^2$ of $(\mathbb{P}^1)^n$. Let $(\alpha, \beta) \in C(\bar{K}) \cap V(\bar{K})$. From $f^k(\alpha) = g(f^k(\beta))$ and Lemma 3.6, we still have $\hat{h}(\alpha) = \deg(g)\hat{\beta}$. Therefore inequality (29) still holds. We still have that $h(\alpha, \beta)$ is bounded when $\deg(g)$ is sufficiently large. Since there are only finitely many g's of bounded degrees (see Proposition 2.14), one may assume that the periodic hypersurface $\{x_j = g(x_i)\}$ is fixed. Our discussions so far implies that Question 3.7(b) is equivalent to the following:

Question 3.8. *Let f, φ and C be as in Question 3.7(b). Let W be a fixed irreducible periodic hypersurface of $(\mathbb{P}^1_{\bar{K}})^n$. For $k \geq 0$, write $\varphi^{-k}(W)$ to denote $(\varphi^k)^{-1}(W)$. Is it true that points in $\bigcup_{k\geq 0} C(\bar{K}) \cap \varphi^{-k}(W)(\bar{K})$ have bounded heights?*

For the rest of this paper, we will focus on Question 3.8. We could only prove a weaker result, namely points in $C(\bar{K}) \cap \varphi^{-k}(W)(\bar{K})$ have bounded "average heights" independent of k (see Subsection 3.5). Such result is motivated by examples given in the next subsection.

3.4. **Examples.** Let f, φ, W and C be as in Question 3.8. We may assume $n = 2$ and W is given by $y = g(x)$ where g commutes with an iterate of f. In this subsection, we look at the case when C is a rational curve parametrized by $(P(t), Q(t))$ where P and Q are polynomials with coefficients in $\bar{K}$. Question 3.8 asks whether roots of $f^k \circ Q = g \circ f^k \circ P$ have heights bounded independently of k. Note that if α is such a root then $\hat{h}(Q(\alpha)) = \deg(g)\hat{h}(P(\alpha))$ by Lemma 3.6. Therefore $|\deg(Q) - \deg(g)\deg(P)|h(\alpha)$ is bounded independently of k. Hence if $\deg(g)\deg(P) \neq \deg(Q)$ then Question 3.8 has an affirmative answer. For the rest of this subsection, we may assume $\deg(g)\deg(P) = \deg(Q)$.

Since g commutes with an iterate f^l of f, we may look at l collections of equations of the form

$$f^{ql+r} \circ Q = g \circ f^{ql+r} \circ P = f^{ql} \circ g \circ f^r \circ P \quad \text{for } 0 \leq q,$$

for each $0 \leq r < l$. Replacing (P, Q) by $(g \circ f^r \circ P, f^r \circ Q)$, we may assume $g(x) = x$ (i.e. V_0 is the diagonal), and hence $\deg(P) = \deg(Q)$. For every $k \geq 0$, put $G_k = f^k \circ P - f^k \circ Q$. We need to show that roots of G_k have heights bounded

independently of k. By making a linear change, we can assume f has the following form:

$$f(X) = X^d + a_{d-2}X^{d-2} + \ldots + a_0.$$

We have the following:

Lemma 3.9. *The linear automorphism of the Julia set of f is a cyclic group $\mathscr{G}(f)$ of order M. Let $D \in \mathbb{Z}$ such that $f \circ L = L^D \circ f$ for every $L \in \mathscr{G}(f)$. If ζ_1 and ζ_2 are two roots of unity such that $f(\zeta_1 X) = \zeta_2 f(X)$ then $\zeta_1 \in \mathscr{G}(f)$, and $\zeta_2 = \zeta_1^D$.*

Proof. This is a classical result in complex dynamics, see [Bea90] and [SS95]. □

Proposition 3.10. (a) *There is a constant c_3 such that*

$$\deg(G_k) \geq c_3 d^k \quad \forall k.$$

(b) *There is a constant c_4 such that the affine height of G_k (*[HS00, Part B]*) satisfies:*

$$h(G_k) \leq c_4 d^k \quad \forall k.$$

(c) *The average height of the roots of G_k are bounded independently of k: there is c_5 such that:*

$$\frac{1}{\deg(G_k)} \sum_{G_k(\alpha)=0} h(\alpha) \leq c_5 \quad \forall k,$$

where we allow repeated roots to appear multiple times in $\sum$.

Proof. For part (a): if there are $k_1 < k_2$ and roots of unity ζ_1, ζ_2 such that:

$$f^{k_i} \circ Q = \zeta_i f^{k_i} \circ P \text{ for } i = 1, 2,$$

then we have that

$$f^{k_2-k_1}(\zeta_1 X) = \zeta_2 f^{k_2-k_1}(X).$$

By Lemma 3.9, we have that $\zeta_1 \in \mathscr{G}(f)$. Hence the curve $\varphi^{k_1}(C)$ which is given by $y = \zeta_1 x$ is preperiodic, contradiction. Therefore, there exists μ such that $\dfrac{f^k \circ Q}{f^k \circ P}$ is not a root of unity for $k \geq \mu$. Write:

$$f^{k-\mu}(X) = X^{d^{k-\mu}} + b_{d^{k-\mu}-2} X^{d^{k-\mu}-2} + \ldots$$

We have (this is the only place where we do not follow our convention on notation: N in the exponent means the usual "raising to the N^{th} power" instead of "taking the N^{th} iterate"):

$$(f^\mu \circ Q)^N - (f^\mu \circ P)^N = \prod_{\zeta^N=1} (f^\mu \circ Q - \zeta f^\mu \circ P).$$

Since at most one factor has degree lower than $d^\mu \deg(P)$, and that factor is a nonzero polynomial, we have:

$$\deg((f^\mu \circ Q)^N - (f^\mu \circ P)^N) \geq (N-1) d^\mu \deg(P).$$

Therefore:

$$\deg(G_k) = \deg(f^{k-\mu} \circ f^\mu \circ Q - f^{k-\mu} \circ f^\mu \circ Q) \geq (d^{k-\mu} - 1) d^\mu \deg(P).$$

This finishes part (a).

For part (b), it suffices to show there are constants ϵ_1 and ϵ_2 such that $h(f^k \circ P) \leq \epsilon_1 d^k$ and $h(f^k \circ Q) \leq \epsilon_2 d^k$ for every k. By similarity, we only need to prove the

existence of ϵ_1. Let $r_1, ..., r_{d^k}$ denote the roots of f^k. Since $\hat{h}_f(r_i) = \dfrac{\hat{h}_f(0)}{d^k}$, we have:

$$\sum_{i=1}^{d^k} h(r_i) = \hat{h}_f(0) + O(1)d^k \tag{30}$$

where $O(1)$ only depends on f (since we change from canonical height to Weil height). From $f^k \circ P = \displaystyle\prod_{i=1}^{d^k}(P - r_i)$, and [HS00, Proposition B.7.2] we have:

$$\begin{aligned} h(f^k \circ P) &\leq \sum_{i=1}^{d^k}(h(P - r_i) + (\deg(P) + 1)\log 2) \\ &\leq \sum_{i=1}^{d^k}(h(P) + h(r_i) + (\deg(P) + 2)\log 2) \\ &= \hat{h}_f(0) + d^k(h(P) + (\deg(P) + 2)\log 2 + O(1)) \end{aligned}$$

where the last equality follows from (30), so the error term $O(1)$ only depends on f. Finally part (c) follows from part (a), part (b) and [BG06, Theorem 1.6.13]. □

Part (c) of Proposition 3.10 only gives us an upper bound (independent of k) for the average of the heights of roots of G_k instead of the height of every root. Now suppose there is a constant c_6 (independent of k) such that for every k, every root α of G_k that is not a root of G_{k-1} has degree at least $c_6 d^k$ over K then we are done. The reason is that there are at least $c_6 d^k$ conjugates of α and all contribute the same height to the average. It is usually the case in the dynamics of disintegrated f that every irreducible factor (in $K[X]$) of G_k has a large degree unless it has already been a factor of G_{k-1}. However, while such phenomena appear in practice, it seems to be a very difficult problem to prove that such lower bounds on the degrees hold in general. We conclude this subsection by cooking up a specific instance in which all irreducible factors of $\dfrac{G_k}{G_{k-1}}$ have large degrees thanks to the Eisenstein's criterion.

Proposition 3.11. *Let $d \geq 2$ and let $p > d$ be a prime. Let $f(X) = X^d + p$, and C be the curve $y = x + p$ in $(\mathbb{P}^1_{\mathbb{Q}})^2$. Then C is non-preperiodic and points in $\bigcup_V C(\bar{K}) \cap V(\bar{K})$ have bounded heights, where V ranges over all preperiodic curves of $(\mathbb{P}^1_{\mathbb{Q}})^2$.*

Proof. By Theorem 2.13 and Proposition 2.14, non-preperiodicity of C is equivalent to $f^k(X) \neq \zeta f^k(X+p)$ for every k, and this is obvious. Hence C is non-preperiodic.

We have:

$$G_k = f^k(x+p) - f^k(x) = \prod_{\zeta^d=1}(f^{k-1}(x+p) - \zeta f^{k-1}(x)).$$

By the reduction from part (b) of Question 3.7 to Question 3.8, it suffices to show that for every periodic W, points in $\bigcup_{k\geq 0} C(\bar{K}) \cap \varphi^{-k}(W)(\bar{K})$ have bounded heights. By the argument in the beginning of this subsection, we may assume W is the diagonal. Hence it suffices to show that roots of G_k have bounded heights independent of k. By Eisenstein's criterion, $f^{k-1}(x+p) - \zeta f^{k-1}(x)$ is irreducible

(over $\mathbb{Q}(\zeta)$) when $\zeta \neq 1$. Then by Proposition 3.10 and the discussion after it, we get the desired conclusion. $\square$

3.5. **The Bounded Average Height Theorem.**

3.5.1. *The Statements.* In this subsection, we prove that the average bounded height result in Proposition 3.10 holds for an arbitrary polarized dynamical system (see Theorem 3.13). We have the following:

Theorem 3.12. *Let f, n and φ be as in Theorem 3.4. Let C be an irreducible curve in $(\mathbb{P}^1_{\bar{K}})^n$ such that its projection to each factor $\mathbb{P}^1_{\bar{K}}$ is onto. There exists a constant c_7 such that for every irreducible preperiodic hypersurface V in $(\mathbb{P}^1_{\bar{K}})^n$ that does not contain C, the average height of points in $C(\bar{K}) \cap V(\bar{K})$ is bounded above by c_7. More precisely, define:*

$$C_{\bar{K}}.V = m_1 P_1 + \ldots + m_l P_l$$

where $C(\bar{K}) \cap V(\bar{K}) = \{P_1, \ldots, P_l\}$ and $m_1, \ldots, m_l$ are the corresponding intersection multiplicities. Then we have:

$$\frac{\sum_{i=1}^{l} m_i h(P_i)}{\sum_{i=1}^{l} m_i} \leq c_7. \tag{31}$$

As in the proof of Theorem 3.4, we can simply reduce to the case $n = 2$. Then Theorem 3.12 is a special case of the following:

Theorem 3.13. *Let X be a projective scheme over K such that $X_{\bar{K}}$ is normal and irreducible, H a closed subscheme of X such that $H_{\bar{K}}$ is an irreducible hypersurface. Assume the line bundle L associated to H is very ample. Let $d \geq 2$, and let φ be a K-morphism from X to itself such that $\varphi^* L \cong L^d$. Fix a height $\tilde{h}$ on $X(\bar{K})$ corresponding to a very ample line bundle. There exists c_8 such that for every irreducible φ-preperiodic curve V of $X_{\bar{K}}$ not contained in $H_{\bar{K}}$, the average height of points in $H(\bar{K}) \cap V(\bar{K})$ is bounded above by c_8. More precisely, write:*

$$H_{\bar{K}}.V_{\bar{K}} = m_1 P_1 + \ldots + m_2 P_2$$

where $H(\bar{K}) \cap V(\bar{K}) = \{P_1, \ldots, P_l\}$ and $m_1, \ldots, m_l$ are the corresponding multiplicities. Then we have:

$$\frac{\sum_{i=1}^{l} m_i \tilde{h}(P_i)}{\sum_{i=1}^{l} m_i} \leq c_8. \tag{32}$$

From now on, we focus on proving Theorem 3.13. Note the amusing change that we now concentrate on the intersection of a fixed *hypersurface* with an arbitrary preperiodic *curve*. We regard X as a closed subvariety of $\mathbb{P}^N_K$ by choosing a closed embedding associated to H. Let h denote the Weil height on $\mathbb{P}^N(\bar{K})$ as well as its restriction on $X(\bar{K})$. We may prove Theorem 3.13 with $\tilde{h}$ replaced by h since there exists M such that $\tilde{h} < Mh + O(1)$ where the error term $O(1)$ is uniform on $X(\bar{K})$. The main ingredients of the proof of Theorem 3.13 are the arithmetic Bézout's theorem by Bost-Gillet-Soulé [BGS94], and the construction of the canonical height for subvarieties by Zhang [Zha95] and Kawaguchi [Kaw06].

3.5.2. *Proof of Theorem 3.13.* Let V be a φ-preperiodic curve in $X_{\bar{K}}$. Let F be a finite extension of K such that V is defined over F. Write $\mathscr{O} = \mathscr{O}_K$ to denote the ring of integers of K, and π to denote the base change morphism $\mathbb{P}^N_F \longrightarrow \mathbb{P}^N_K$. As in [BGS94, p.946–947], we let $\bar{E}$ denote the trivial hermitian vector bundle of rank $N+1$ on $\mathrm{Spec}(\mathscr{O})$ and equip the canonical line bundle $\mathscr{M} := \mathscr{O}(1)$ of $\mathbb{P}^N_{\mathscr{O}}$ with the quotient metric m. We denote $\bar{\mathscr{M}} = (\mathscr{M}, m)$. The pull-back of $\mathscr{M}$ to X is isomorphic to the line bundle L.

For $0 \leq p \leq N+1$, for any cycle $\mathscr{Z} \in Z_p(\mathbb{P}^N_{\mathscr{O}})$ of dimension p, following [BGS94, p. 946], we define the Faltings' height of $\mathscr{Z}$ to be the real number:

$$h_{Fal}(\mathscr{Z}) = \widehat{\deg}\left(\hat{c}_1(\bar{\mathscr{M}})^p \mid \mathscr{Z}\right) \tag{33}$$

where $\hat{c}_1(\bar{\mathscr{M}})$ is the first arithmetic Chern class of $\bar{\mathscr{M}}$, and $\widehat{\deg}$ is the arithmetic degree map as defined in [BGS94].

For $0 \leq p \leq N$, for every cycle $Z \in Z_p(\mathbb{P}^N_K)$, let $\bar{Z}$ denote the closure of Z in $\mathbb{P}^N_{\mathscr{O}}$. We define the Faltings' height of Z to be:

$$h_{Fal}(Z) := h_{Fal}(\bar{Z}). \tag{34}$$

If $Z \in Z_p(\mathbb{P}^N_{\bar{K}})$, we let K' be a finite extension of K so that Z is defined over K', i.e. Z is the pull-back of a cycle $Z' \in Z_p(\mathbb{P}^N_{K'})$. Let ρ denote the base change morphism from $\mathbb{P}^N_{K'}$ to $\mathbb{P}^N_K$. We then define the Faltings' height of Z to be:

$$h_{Fal}(Z) := \frac{1}{[K':K]} h_{Fal}(\rho_* Z') \tag{35}$$

This is independent of the choice of K'.

For $0 \leq p \leq N+1$, for every cycle $\mathscr{Z} \in Z_p(\mathbb{P}^N_{\mathscr{O}})$, we have the following Bost-Gillet-Soulé projective height of $\mathscr{Z}$ [BGS94, p. 964]:

$$h_{BGS}(\mathscr{Z}) = \widehat{\deg}(\hat{c}_p(\bar{Q}) \mid \mathscr{Z}) \tag{36}$$

where $\bar{Q}$ is the hermitian vector bundle defined as in [BGS94, p. 964], and $\hat{c}_p$ is the p^{th} arithmetic Chern class of $\bar{Q}$.

For $0 \leq p \leq N$, for every cycle $Z \in Z_p(\mathbb{P}^N_K)$, we define the Bost-Gillet-Soulé height of Z to be:

$$h_{BGS}(Z) := h_{BGS}(\bar{Z}). \tag{37}$$

If $Z \in Z_p(\mathbb{P}^N_{\bar{K}})$, we let K' be a finite extension of K over which Z is defined by $Z' \in Z_p(\mathbb{P}^N_{K'})$. Let ρ be the base change morphism as above, we define:

$$h_{BGS}(Z) := \frac{1}{[K':K]} h_{BGS}(\rho_* Z'). \tag{38}$$

This is independent of the choice of K'.

Proposition 4.1.2 in [BGS94] in which the authors compare the Faltings' height and the Bost-Gillet-Soulé projective height yields the following:

Proposition 3.14. *For $0 \leq p \leq N$, for any cycle $Z \in Z_p(\mathbb{P}^N_K)$, define $\deg_K(Z) = \deg_K(\bar{Z}) := \deg_{\mathscr{O}(1)_K}(Z)$ as in* [BGS94, p. 964]. *We have:*

$$h_{BGS}(Z) = h_{Fal}(Z) - [K:\mathbb{Q}]\sigma_p \deg_K(Z), \tag{39}$$

where σ_p is the Stoll number (see, for example, [BGS94, p. 922]*).*

The arithmetic Bézout theorem [BGS94, Theorem 4.2.3] implies the following:

Proposition 3.15. *Let $Y \in Z_{N-1}(\mathbb{P}^N_K)$ and $Z \in Z_1(\mathbb{P}^N_K)$ be two cycles intersecting properly in $\mathbb{P}^N_K$. We have:*

$$\begin{aligned} h_{BGS}(Y.Z) \leq\ & \deg_K(Z) h_{BGS}(Y) + h_{BGS}(Z)\deg_K(Y) \\ & + [K:\mathbb{Q}]a(N,N,2)\deg_K(Y)\deg_K(Z) \end{aligned} \tag{40}$$

where $a(N,N,2)$ is the constant defined in [BGS94, p. 971].

To prove Proposition 3.15, note the following:

$$h_{BGS}(Y.Z) := h_{BGS}(\overline{Y.Z}) \leq h_{BGS}(\bar{Y}.\bar{Z})$$

because the closure $\overline{Y.Z}$ of $Y.Z$ is contained in $\bar{Y}.\bar{Z}$. Then we bound $h_{BGS}(\bar{Y}.\bar{Z})$ from above by the right hand side of (40) thanks to [BGS94, Theorem 4.2.3].

Let H' denote the hyperplane of $\mathbb{P}^N_K$ whose restriction to X is H. Define $V' = \dfrac{V}{\sum_{i=1}^{l} m_i}$ as a pure cycle (with rational coefficients) in $\mathbb{P}^N_F$. By the classical Bézout's theorem, we have:

$$\begin{aligned} \deg_K(\pi_* V') = [F:K]\deg_F(V') &= \frac{[F:K]\deg_F(V)}{\deg_{\bar{K}}(H'_{\bar{K}}.V_{\bar{K}})} \\ &= \frac{[F:K]\deg_F(V)}{\deg_F(H'_F.V)} = \frac{[F:K]}{\deg_F(H'_F)} = [F:K]. \end{aligned} \tag{41}$$

$$\deg_K(H'.\pi_* V') = [F:K]. \tag{42}$$

Apply Proposition 3.15 for the cycles H' and $\pi_* V'$ together with (41) and (42), we have:

$$h_{BGS}(H'.\pi_* V') \leq [F:K]h_{BGS}(H') + h_{BGS}(\pi_* V') + [F:\mathbb{Q}]a(N,N,2). \tag{43}$$

By using Proposition 3.14, (41), (42) and the fact that $\sigma_0 = 0$, we can replace h_{BGS} by h_{Fal} in (43) to get:

$$\begin{aligned} h_{Fal}(H'.\pi_* V') \leq [F:K](h_{Fal}(H') - [K:\mathbb{Q}]\sigma_{N-1}) + h_{Fal}(\pi_* V') \\ - [F:\mathbb{Q}]\sigma_1 + [F:\mathbb{Q}]a(N,N,2). \end{aligned} \tag{44}$$

Therefore

$$h_{Fal}(H'.\pi_* V') \leq [F:K]h_{Fal}(H') + h_{Fal}(\pi_* V') + [F:\mathbb{Q}]c_{10}, \tag{45}$$

where $c_{10} = a(N,N,2) - \sigma_1 - \sigma_{N-1}$ is an explicit constant depending only on N.

Dividing both sides of (45) by $[F:\mathbb{Q}]$, we have:

$$\frac{h_{Fal}(H'.\pi_* V')}{[F:\mathbb{Q}]} \leq \frac{h_{Fal}(H')}{[K:\mathbb{Q}]} + \frac{h_{Fal}(V')}{[K:\mathbb{Q}]} + c_{10} \tag{46}$$

From $H'_{\bar{K}}.V'_{\bar{K}} = \dfrac{\sum_{i=1}^{l} m_i P_i}{\sum_{i=1}^{l} m_i}$, we have:

$$\frac{h_{Fal}(H'.\pi_* V')}{[F:K]} = \frac{\sum_{i=1}^{l} m_i h_{Fal}(P_i)}{\sum_{i=1}^{l} m_i} \tag{47}$$

Recall that h denote the absolute Weil height on $\mathbb{P}^N(\bar{K})$ (see the paragraph right after Theorem 3.13). Note that h_{Fal} on $\mathbb{P}^N(\bar{K})$ is also a choice of a Weil height

(relative over K) corresponding the canonical line bundle $\mathscr{O}(1)$. Hence there exists a constant c_{11} such that:

$$|h(P) - \frac{h_{Fal}(P)}{[K:\mathbb{Q}]}| \leq c_{11} \ \forall P \in \mathbb{P}^N(\bar{K}). \tag{48}$$

From (46), (47) and (48), we have:

$$\frac{\sum_{i=1}^{l} m_i h(P_i)}{\sum_{i-1}^{l} m_i} \leq \frac{h_{Fal}(V')}{[K:\mathbb{Q}]} + \frac{h_{Fal}(H')}{[K:\mathbb{Q}]} + c_{10} + c_{11}.$$

To finish the proof of Theorem 3.13, it remains to show that $\dfrac{h_{Fal}(V')}{[K:\mathbb{Q}]}$ is bounded independently of V. We will use the canonical height $h_{\varphi,L}$ constructed by Zhang [Zha95] and generalized by Kawaguchi [Kaw06]. We have the following special case of their construction:

Proposition 3.16. *There is a height function $h_{\varphi,L}$ on effective cycles in $Z_1(X_{\bar{K}})$ satisfying the following properties:*

(a) *If Z is a preperiodic curve in $X_{\bar{K}}$ then $h_{\varphi,L}(Z) = 0$.*
(b) *There exists a constant c_{12} such that for every curve Z in $X_{\bar{K}}$, we have:*

$$|h_{\varphi,L}(Z) - \frac{h_{Fal}(Z)}{2[K:\mathbb{Q}]\deg_{\bar{K}}(Z)}| < c_{12}$$

Part (a) follows from [Zha95, Theorem 2.4], and part (b) follows from [Kaw06, Theorem 2.3.1]. The preperiodicity of V together with Proposition 3.16 yield:

$$\frac{h_{Fal}(V')}{[K:\mathbb{Q}]} = \frac{h_{Fal}(V)}{[K:\mathbb{Q}]\deg_{\bar{K}}(V)} < 2c_{12}$$

which finishes the proof of Theorem 3.13.

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